Applied Econometrics – GARCH Models

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Lecture 1

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Motivation

- Natural relation between return and volatility (risk), CAPM
- Pricing of options based on the volatility of asset returns

How to measure volatility?

- Historical volatility
  - » Sample standard deviation of returns over a short period
  - » What is the right period? This is chosen rather arbitrarily
  - » Equal weights to yesterday and before 3-weeks observations?
  - » This is historical volatility, but we need expected volatility
  - » Volatility is not time-varying, this is at odds with stylized facts
- Exponentially weighted moving average models
  - » Greater weight to yesterday than before 3-weeks observations, weight exponentially declining
  - » But still rather arbitrary and backward-looking
- GARCH

Stylized facts in finance

- Unpredictability
  - Efficient markets
- Volatility
  - Time-varying (turbulent vs. tranquil periods)
  - Volatility clustering
    - » When volatility is high, likely to remain high
    - » Clustering of information arrivals, price discovery
- Fat tails
  - Extremely large returns (both positive and negative) more likely than standard normal distribution

ARCH – GARCH family

- ARCH = autoregressive conditional heteroscedasticity
- GARCH = generalized autoregressive conditional heteroscedasticity
- Dozens of GARCH models, for example:
    - EGARCH – exponential GARCH,
    - FIGARCH – fractionally integrated GARCH,
    - TARCH – threshold GARCH

Introduction

- Analysis and forecasting of the size of errors and its development over time
- Squared residuals often autocorrelated even though residuals themselves are not
- ARCH/GARCH for modelling volatility of asset – wide application in finance
- OLS assumes homoscedasticity, here we analyze heteroscedasticity, namely conditional heteroscedasticity (conditional on past information, as opposed to unconditional heteroscedasticity)

Introduction

- What does GARCH do?
- Imagine you own some asset and would like to know the tomorrow volatility of asset to evaluate its risk
- You may arbitrarily take last X observations (say 22 days) and take standard deviation. In this case, you assign the same weight to yesterday's observation and observation before tree weeks, which is unrealistic
- Alternatively, use GARCH

Introduction continued

- GARCH ‘says’ that best predictor of variance for today is a weighted average of:
    - the long-run variance (constant),
    - the forecast made in previous period and,
    - the new information that was not available when previous forecast was made (captured by the most recent squared residual)

So far concerned about long run statistics - notes

- Long-run moments of series – remember for example as we showed that forecast for ARMA model converges to its long-run mean, but you might be interested, what is mean likely to be in the next period
- Useful to distinguish between unconditional and conditional moments of series, unconditional - those of long-run, conditional – those of actual values of series in previous periods

- Assume that a stationary series of interest, y_t, follows simply:
- $y_t = \varrho y_{t-1} + u_t$
  - Unconditional mean is $E(y_t) = 0$
  - Unconditional variance is $Var(y_t)=\sigma^2/(1-\varrho^2)$
  - Conditional mean is $E(y_t|\Omega_{t-1}) = \varrho y_{t-1}$
  - Conditional variance $Var(y_t|\Omega_{t-1})=\sigma_t^2$
  - Where Ω_{t-1} is information set, e.g. all what is known before t
- Note that conditional mean is time-varying and conditional variance is *smaller* than unconditional variance
- $y_t = \varrho y_{t-1} + u_t$ with such equation you already know how to predict mean, how about predicting variance of y_t?

ARCH

- Note that $E(y_t)^2 = E(\varrho y_{t-1} + u_t - E(y_t))^2 = E(\varrho y_{t-1} + u_t - \varrho y_{t-1})^2 = E(u_t)^2$

  => for predicting var(y_t) it is enough to predict var(u_t)!

- Engle (1982) proposed following model for the error term u_t: $u_t = e_t(a_0 + a_1 * u_{t-1}^2)^{1/2}$
- Assume also that $a_0 > 0$ and $0 < a_1 < 1$, $e_t \sim IID(0,1)$

ARCH

- Note that conditional $E(u_t|\Omega_{t-1})=0$ and conditional $\text{var}(u_t|\Omega_{t-1}) = E(u_t^2|\Omega_{t-1}) = a_0 + a_1 * u_{t-1}^2$ – so conditional variance is not constant over time!
- Now, going back to our initial model $y_t = \varrho y_{t-1} + u_t$
- So, conditional mean of y_t is ϱy_{t-1} and conditional variance of y_t is $E(y_t)^2 = E(u_t)^2 = a_0 + a_1 * u_{t-1}^2$

ARCH

- We just derived ARCH(1) model
- ARCH says that variability of y_t in the next period is a function of this period squared residual and constant
- After estimating α_0 and α_1, we can predict the variance for the next period
    - ARCH straightforward to apply in empirical finance
        - Basic pricing models in finance: if you know that y_t and x_t provides same return, but you predict greater volatility in the next period for x_t, which one are you going to hold?

ARCH – "2 equations model"

- Mean equation (example – AR(1)):
  - $y_t = \varrho y_{t-1} + u_t$
- Variance equation:
  - $E(u_t)^2 = a_0 + a_1 * u_{t-1}^2$
- This is to be estimated jointly be maximum likelihood
- In principle, one can also estimate it equation by equation by OLS, but this is less efficient
- Again, all specification checks to be addressed like in ARIMA modeling
    - Such as are residuals white noise?

ARCH(q)

- ARCH (1) simply generalizes into ARCH (q), q specifies the number of lagged squared error terms
- $u_t = e_t(a_0 + a_1 {*} u_{t-1}^2 + \ldots . + a_q {*} u_{t-q}^2)^{1/2}$
- Then, Var(y_t) = $a_0 + a_1 {*} u_{t-1}^2 + \ldots . + a_q {*} u_{t-q}^2$
- You may include dummies or any variable into this equation:
  - say day of the week dummy, deviation of exchange rate from central parity, etc.

Problem with ARCH

- Sometimes large number of squared lagged residuals must be included to specify the model correctly
- Bolerslev (1986) extends ARCH model to allow more flexible lag structure – he introduces GARCH, it is to some extent analogy to ARMA models – ARCH is like MA model and now we introduce ‘AR’

GARCH (1,1)

- Note that $E(y_t)^2 = E(u_t)^2 = a_0 + a_1 {*} u_{t-1}{}^2$
- Include lags of $E(u_t)^2$ to the RHS of the equation $E(u_t)^2 = a_0 + a_1 {*} u_{t-1}{}^2 + \beta_1 {*} E(u_{t-1})^2$
- Typically, we write it this way:
- $h_t = a_0 + a_1 {*} u_{t-1}{}^2 + \beta_1 {*} h_{t-1}$
- So now, what is the 'best' prediction for the next period variance of y_t? Weighted average (with the weights in brackets) of:
  - long-term variance (a_0),
  - this period actual variance – you may call it new information, not captured anywhere else (a_1),
  - the variance predicted for this period (β_1)

GARCH (p,q)

- Naturally, you may generalize GARCH(1,1) to GARCH(p,q) assuming this model for residual
- $u_t = e_t(a_0 + a_1 * u_{t-1}{}^2 + \ldots. + a_q * u_{t-q}{}^2 + \beta_1 * h_{t-1} + \ldots. + \beta_p * h_{t-p})^{1/2}$
- Note that e_t ~IID(0,1), $a_0 > 0$, $a_1 \geq 0, \ldots, a_q \geq 0$, $q > 0$, $p \geq 0$, $\beta_1 \geq 0$, …, $\beta_p \geq 0$
- GARCH (p,q) is stationary, if the sum of *a's* all and β's is strictly smaller than 1
- Conditional variance $E(y_t)^2 = E(u_t)^2 = h_t$
- $= a_0 + a_1 * u_{t-1}{}^2 + \ldots. + a_q * u_{t-q}{}^2 + \beta_1 * h_{t-1} + \ldots. + \beta_p * h_{t-p}$

How to estimate GARCH - strategy

1. Fit correct ARIMA model for y_t, and check if residuals are white noise etc.
2. If you suspect (G)ARCH residuals, estimate $u_t^2 = a_0 + a_1 * u_{t-1}^2 + \ldots + a_q * u_{t-q}^2 + v_t$ (simply regress residual on its lags, test significance, ARCH-LM test)
3. If ARCH errors not present, all explanatory variables should be jointly insignificant from zero
4. If jointly significant, this tells you that is some ARCH or GARCH structure of residuals
5. Examine PACF and ACF of squared residuals, reasoning same as for ARIMA modelling, (but note that very often you find that GARCH(1,1) fits the data best)
6. Evaluate the model:
    - Are residuals white noise?

Extensions: TARCH

- Leverage effect in finance:
    - Bad news are more important than good news for the behavior of stock
    - In our case, bad news = negative residual (u_t<0)
    - Good news = positive residual (u_t>0)
- TARCH accounts for this effect
    - It includes "dummy*bad news"
- Thus, TARCH (1,1):
    - $y_t = \varrho y_{t-1} + u_t$
- $h_t = a_0 + a_1 * u_{t-1}^2 + \beta_1 * h_{t-1} + \gamma_1 * I_{t-1} * u_{t-1}^2$
    - Where I_t =1, if u_t<0 and 0 otherwise
- GARCH is thus special case of TARCH
- If γ_1=0, no asymmetric effects ("GARCH=TARCH")

Readings

- <u>Theory:</u>
  – Engle, R. (2001): GARCH 101: The Use of ARCH/GARCH Models in Applied Econometrics, *Journal of Economic Perspectives*, pp. 157-168.
  – Engle, R. (2003): Risk and Volatility: Econometric Models and Financial Practice, Nobel lecture, December 8, 2003,
  – The Royal Swedish Academy of Sciences (2003): Time Series Econometrics: Cointegration and Autoregressive Conditional Heteroscedasticity
- <u>Applications:</u>
  – Kocenda, E. (1998): Exchange Rate in Transition (chapters 2-3), CERGE-EI
  – Fidrmuc, J. and R. Horvath (2008): Volatility of Exchange Rates in Selected new EU Members, *Economic Systems*, 103-118.
  – Choudhry, T. (2000): Day of the week effect in emerging Asian stock markets: Evidence from the GARCH model, *Applied Financial Economics*, 235-242.

SEE THE COURSE WEBSITE FOR SOME OF THESE READINGS

- GARCH models are a bit more advanced and are not covered in every introductory econometrics textbook. All financial econometrics textbooks cover it, Brooks (2008) *Introductory Econometrics for Finance* is recommended

The Volatility Laboratory of R. Engle available at
http://vlab.stern.nyu.edu/