

# Applied Econometrics – GARCH Models - Extensions

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Lecture 2

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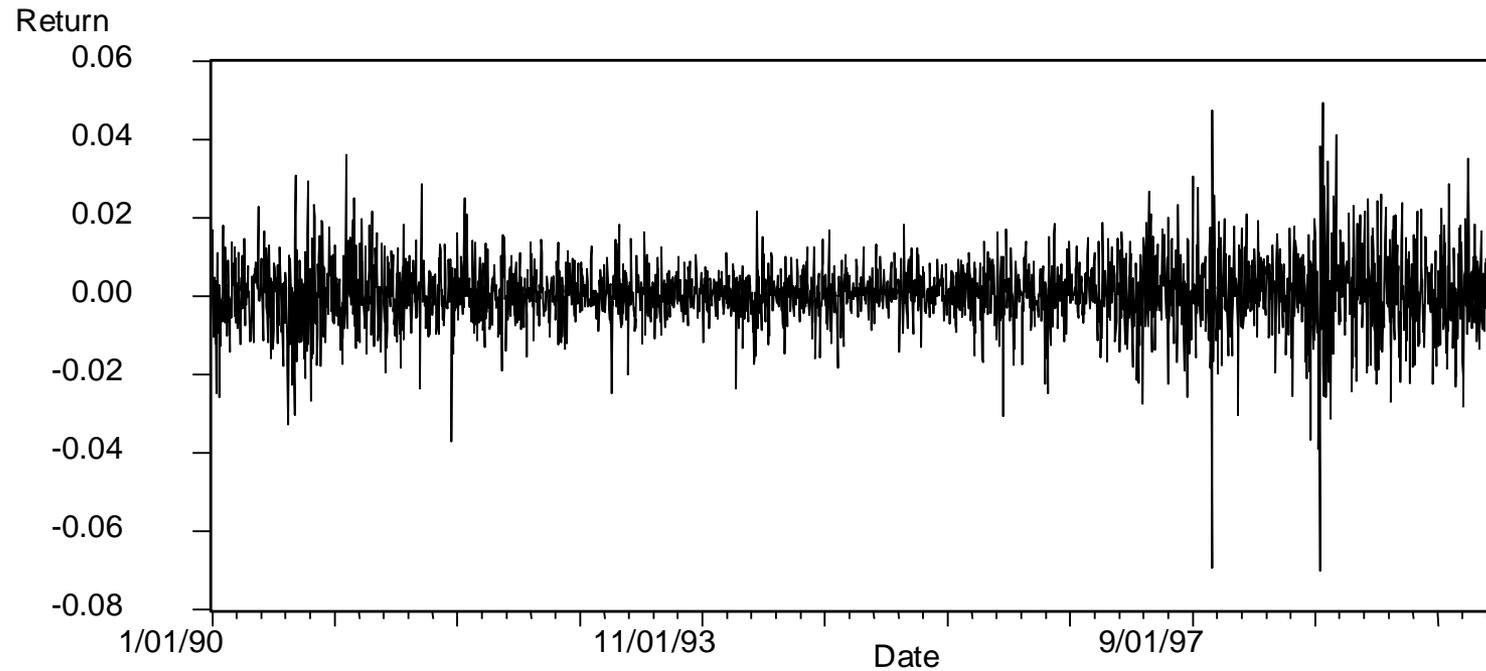
- GARCH
- EGARCH, GARCH-M
- Multivariate GARCH

# Stylized facts in finance

- Unpredictability
  - Efficient markets
- Volatility
  - Time-varying (turbulent vs. tranquil periods)
  - Volatility clustering
    - » When volatility is high, likely to remain high
    - » Clustering of information arrivals, price discovery
- Fat tails
  - Extremely large returns (both positive and negative) more likely than standard normal distribution

# A Sample Financial Asset Returns Time Series

**Daily S&P 500 Returns for January 1990 – December 1999**



# ARCH – “2 equations model”

- Mean equation (example – AR(1)):
  - $y_t = \varrho y_{t-1} + u_t$
- Variance equation:
  - $E(u_t)^2 = a_0 + a_1 * u_{t-1}^2$
- This is to be estimated jointly by maximum likelihood
- In principle, one can also estimate it equation by equation by OLS, but this is less efficient
- Again, all specification checks to be addressed like in ARIMA modeling

# ARCH(q)

- ARCH (1) simply generalizes into ARCH (q), q specifies the number of lagged squared error terms
- $u_t = e_t(a_0 + a_1 * u_{t-1}^2 + \dots + a_q * u_{t-q}^2)^{1/2}$
- Then,  $\text{Var}(y_t) = a_0 + a_1 * u_{t-1}^2 + \dots + a_q * u_{t-q}^2$
- You may include dummies or any variable into this equation:
  - say day of the week dummy, deviation of exchange rate from central parity, etc.

# Problem with ARCH

- Sometimes the large number of squared lagged residuals must be included to specify the model correctly
- Bollerslev (1986) extends ARCH model to allow more flexible lag structure – he introduces GARCH, it is to some extent analogy to ARMA models – ARCH is like MA model and now we introduce ‘AR’

# GARCH (1,1)

- Note that  $E(y_t)^2 = E(u_t)^2 = a_0 + a_1 * u_{t-1}^2$
- Include the lags of  $E(u_t)^2$  to the RHS of the equation  $E(u_t)^2 = a_0 + a_1 * u_{t-1}^2 + \beta_1 * E(u_{t-1})^2$
- Typically, we write it this way:
- $h_t = a_0 + a_1 * u_{t-1}^2 + \beta_1 * h_{t-1}$
- So now, what is the ‘best’ prediction for the next period variance of  $y_t$ ? Weighted average (with the weights in brackets) of:
  - long-term variance ( $a_0$ ),
  - this period actual variance – you may call it new information, not captured anywhere else ( $a_1$ ),
  - the variance predicted for this period ( $\beta_1$ )

# GARCH (p,q)

- Naturally, you may generalize GARCH(1,1) to GARCH(p,q) assuming this model for residual
- $$u_t = e_t(a_0 + a_1 * u_{t-1}^2 + \dots + a_q * u_{t-q}^2 + \beta_1 * h_{t-1} + \dots + \beta_p * h_{t-p})^{1/2}$$
- Note that  $e_t \sim \text{IID}(0,1)$ ,  $a_0 > 0$ ,  $a_1 \geq 0, \dots, a_q \geq 0$ ,  $q > 0$ ,  $p \geq 0$ ,  $\beta_1 \geq 0, \dots, \beta_p \geq 0$
- GARCH (p,q) is stationary, if the sum of  $a$ 's all and  $\beta$ 's is strictly smaller than 1
- Conditional variance  $E(y_t)^2 = E(u_t)^2 = h_t$
- $= a_0 + a_1 * u_{t-1}^2 + \dots + a_q * u_{t-q}^2 + \beta_1 * h_{t-1} + \dots + \beta_p * h_{t-p}$

# How to estimate GARCH - strategy

1. Fit correct ARIMA model for  $y_t$ , and check if residuals are white noise etc.
2. If you suspect (G)ARCH residuals, estimate  $u_t^2 = a_0 + a_1 * u_{t-1}^2 + \dots + a_q * u_{t-q}^2 + v_t$  (simply regress residual on their lags, test significance, ARCH-LM test)
3. If ARCH errors not present, all explanatory variables should be jointly insignificant from zero
4. If jointly significant, this tells you that is some ARCH or GARCH structure of residuals
5. Examine PACF and ACF of squared residuals, reasoning same as for ARIMA modelling, (but note that very often you find that GARCH(1,1) fits the data best)
6. Evaluate the model

# Extensions: TARCh

- Leverage effect in finance:
  - Bad news are more important than good news for the behavior of stock
  - In our case, bad news = negative residual ( $u_t < 0$ )
  - Good news = positive residual ( $u_t > 0$ )
- TARCh accounts for this effect
  - It includes “dummy\*bad news”
- Thus, TARCh (1,1):
  - $y_t = \rho y_{t-1} + u_t$
- $h_t = a_0 + a_1 * u_{t-1}^2 + \beta_1 * h_{t-1} + \gamma_1 * I_{t-1} * u_{t-1}^2$ 
  - Where  $I_t = 1$ , if  $u_t < 0$  and 0 otherwise
- GARCH is thus special case of TARCh
- If  $\gamma_1 = 0$ , no asymmetric effects (“GARCH=TARCh”)
- TARCh sometimes called GJR model

# Exponential GARCH (EGARCH)

- Suggested by Nelson (1991). The variance equation is given by

$$\log(\sigma_t^2) = \omega + \beta \log(\sigma_{t-1}^2) + \gamma \frac{u_{t-1}}{\sqrt{\sigma_{t-1}^2}} + \alpha \left[ \frac{|u_{t-1}|}{\sqrt{\sigma_{t-1}^2}} - \sqrt{\frac{2}{\pi}} \right]$$

- Advantages of the model
  - Since we model the  $\log(\sigma_t^2)$ , then even if the parameters are negative,  $\sigma_t^2$  will be positive.
  - We can account for the leverage effect: if the relationship between volatility and returns is negative,  $\gamma$ , will be negative.

# GARCH-in Mean

- We expect a risk to be compensated by a higher return. So why not let the return of a security be partly determined by its risk?
- Engle, Lilien and Robins (1987) suggested the ARCH-M specification. A GARCH-M model would be

$$y_t = \mu + \delta\sigma_{t-1} + u_t, \quad u_t \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2$$

- $\delta$  can be interpreted as a sort of risk premium.
- It is possible to combine all or some of these models together to get more complex “hybrid” models - e.g. an ARMA-EGARCH(1,1)-M model.

## Forecasting Volatility

- GARCH can model the volatility clustering effect since the conditional variance is autoregressive. Such models can be used to forecast volatility.

- We showed that

$$\text{Var} (y_t \mid y_{t-1}, y_{t-2}, \dots) = \text{Var} (u_t \mid u_{t-1}, u_{t-2}, \dots)$$

- So modelling  $\sigma_t^2$  will give us models and forecasts for  $y_t$  as well.
- Variance forecasts are additive over time.

# Forecasting Variances using GARCH Models

- Producing conditional variance forecasts from GARCH models uses a very similar approach to producing forecasts from ARMA models.
- It is again an exercise in iterating with the conditional expectations operator.

- Consider the following GARCH(1,1) model:

$$y_t = \mu + u_t \quad , \quad u_t \sim \mathbf{N}(0, \sigma_t^2), \quad \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2$$

- What is needed is to generate are forecasts of  $\sigma_{T+1}^2 \mid \Omega_T$ ,  $\sigma_{T+2}^2 \mid \Omega_T, \dots, \sigma_{T+s}^2 \mid \Omega_T$  where  $\Omega_T$  denotes all information available up to and including observation  $T$ .
- Adding one to each of the time subscripts of the above conditional variance equation, and then two, and then three would yield the following equations

$$\sigma_{T+1}^2 = \alpha_0 + \alpha_1 + \beta \sigma_T^2, \quad \sigma_{T+2}^2 = \alpha_0 + \alpha_1 + \beta \sigma_{T+1}^2, \quad \sigma_{T+3}^2 = \alpha_0 + \alpha_1 + \beta \sigma_{T+2}^2$$

# What Use Are Volatility Forecasts?

## 1. Option pricing

Option price is a function of volatility of underlying asset.

## 2. Conditional betas ( $\sigma_{im,t}$ – covariance between market returns and returns of stock $i$ , $\sigma_{m,t}^2$ – variance of market returns at $t$ )

$$\beta_{i,t} = \frac{\sigma_{im,t}}{\sigma_{m,t}^2}$$

## 3. Dynamic hedge ratios

The Hedge Ratio - the size of the futures position to the size of the underlying exposure, i.e. the number of futures contracts to buy or sell per unit of the spot good.

# What Use Are Volatility Forecasts?

- What is the optimal value of the hedge ratio?
- Assuming that the objective of hedging is to minimise the variance of the hedged portfolio, the optimal hedge ratio will be given by

$$h = p \frac{\sigma_S}{\sigma_F}$$

where  $h$  = hedge ratio

$p$  = correlation coefficient between change in spot price ( $S$ ) and change in futures price ( $F$ )

$\sigma_S$  = standard deviation of  $S$

$\sigma_F$  = standard deviation of  $F$

- What if the standard deviations and correlation are changing over time?

Use

$$h_t = p_t \frac{\sigma_{s,t}}{\sigma_{F,t}}$$

## Multivariate GARCH Models

- Multivariate GARCH models are used to estimate and to forecast covariances and correlations. The basic formulation is similar to that of the GARCH model, but where the covariances as well as the variances are permitted to be time-varying.
- There are 3 main classes of multivariate GARCH formulation that are widely used: VECH, diagonal VECH and BEKK.

### VECH and Diagonal VECH

- e.g. suppose that there are two variables used in the model. The conditional covariance matrix is denoted  $H_t$ , and would be  $2 \times 2$ .  $H_t$  and  $\text{VECH}(H_t)$  are

$$H_t = \begin{bmatrix} h_{11t} & h_{12t} \\ h_{21t} & h_{22t} \end{bmatrix} \quad \text{VECH}(H_t) = \begin{bmatrix} h_{11t} \\ h_{22t} \\ h_{12t} \end{bmatrix}$$

# VECH and Diagonal VECH

- In the case of the VECH, the conditional variances and covariances would each depend upon lagged values of all of the variances and covariances and on lags of the squares of both error terms and their cross products.

- In matrix form, it would be written

$$VECH(H_t) = C + A VECH(\Xi_{t-1} \Xi'_{t-1}) + B VECH(H_{t-1}) \quad \Xi_t | \psi_{t-1} \sim N(0, H_t)$$

- Writing out all of the elements gives the 3 equations as

$$h_{11t} = c_{11} + a_{11}u_{1t}^2 + a_{12}u_{2t}^2 + a_{13}u_{1t}u_{2t} + b_{11}h_{11t-1} + b_{12}h_{22t-1} + b_{13}h_{12t-1}$$

$$h_{22t} = c_{21} + a_{21}u_{1t}^2 + a_{22}u_{2t}^2 + a_{23}u_{1t}u_{2t} + b_{21}h_{11t-1} + b_{22}h_{22t-1} + b_{23}h_{12t-1}$$

$$h_{12t} = c_{31} + a_{31}u_{1t}^2 + a_{32}u_{2t}^2 + a_{33}u_{1t}u_{2t} + b_{31}h_{11t-1} + b_{32}h_{22t-1} + b_{33}h_{12t-1}$$

- Such a model would be hard to estimate. The diagonal VECH is much simpler and is specified, in the 2 variable case, as follows:

$$h_{11t} = \alpha_0 + \alpha_1 u_{1t-1}^2 + \alpha_2 h_{11t-1}$$

$$h_{22t} = \beta_0 + \beta_1 u_{2t-1}^2 + \beta_2 h_{22t-1}$$

$$h_{12t} = \gamma_0 + \gamma_1 u_{1t-1} u_{2t-1} + \gamma_2 h_{12t-1}$$

# BEKK and Model Estimation for M-GARCH

- Neither the VECH nor the diagonal VECH ensure a positive definite variance-covariance matrix.
- An alternative approach is the BEKK model (Engle & Kroner, 1995).
- In matrix form, the BEKK model is

$$H_t = W'W + A'H_{t-1}A + B'\Xi_{t-1}\Xi'_{t-1}B$$

- Model estimation for all classes of multivariate GARCH model is again performed using maximum likelihood with the following

*LLF*:

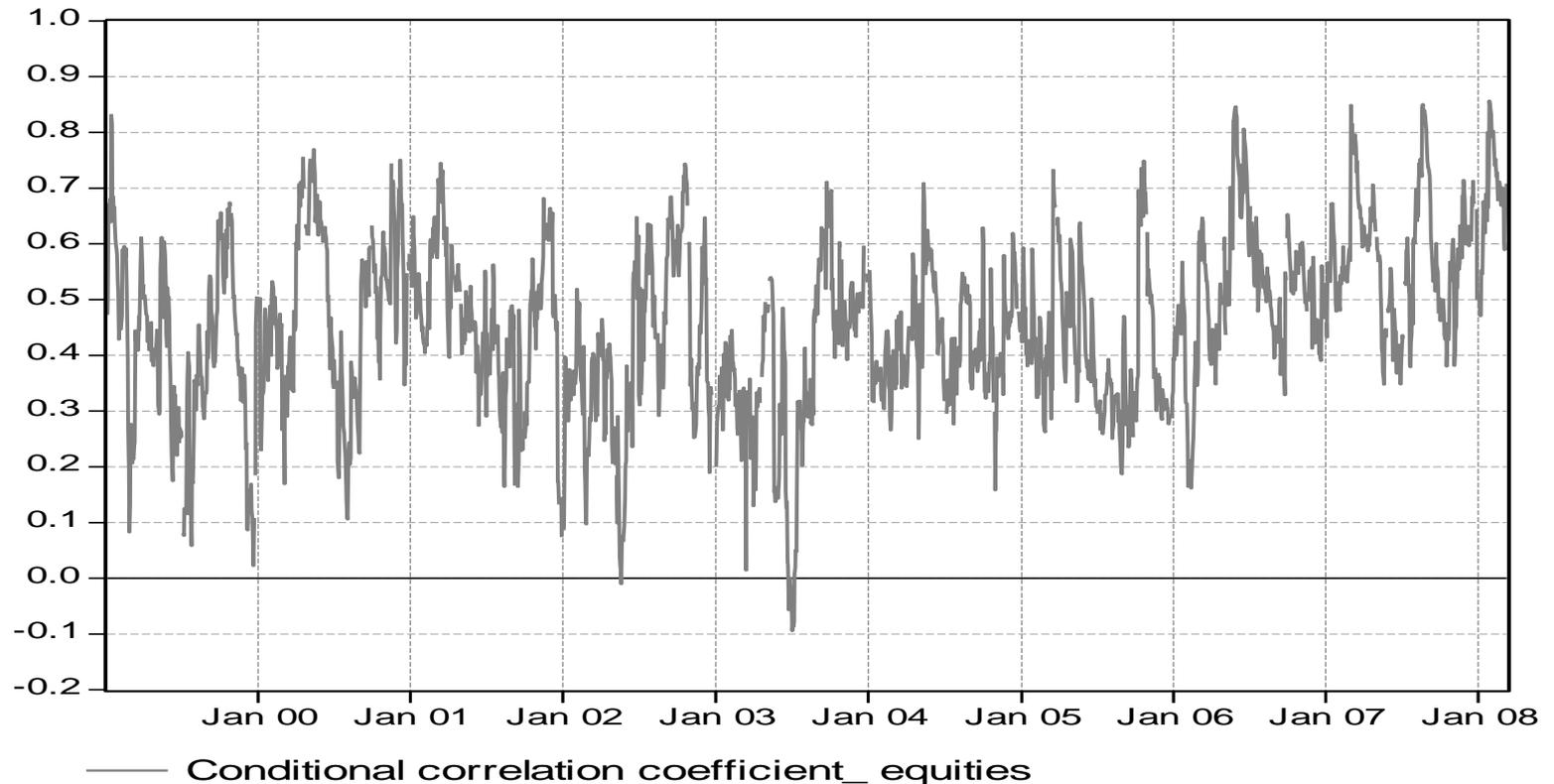
$$\ell(\theta) = -\frac{TN}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^T (\log |H_t| + \Xi_t' H_t^{-1} \Xi_t)$$

where  $N$  is the number of variables in the system (assumed 2 above),  $\theta$  is a vector containing all of the parameters to be estimated, and  $T$  is the number of observations.

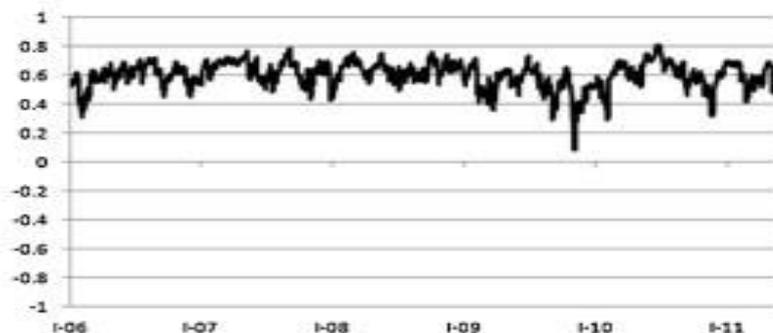
# Conditional correlation between stock markets

M.A. Thesis of Tereza Horáková on Volatility Transmissions between Financial Markets

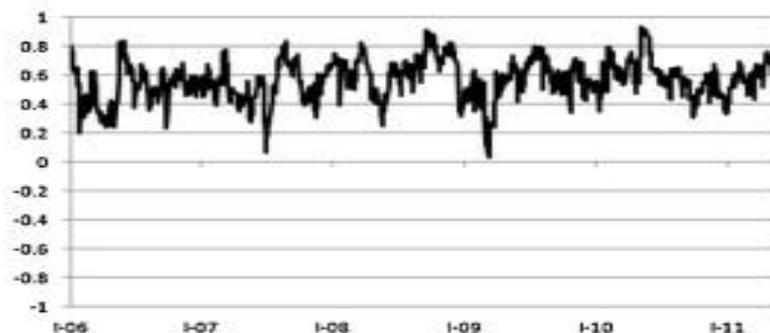
Figure E5: Conditional correlation coefficient for equities market returns ( $r_{PX}$ ,  $r_{DJ}$ ) fluctuations



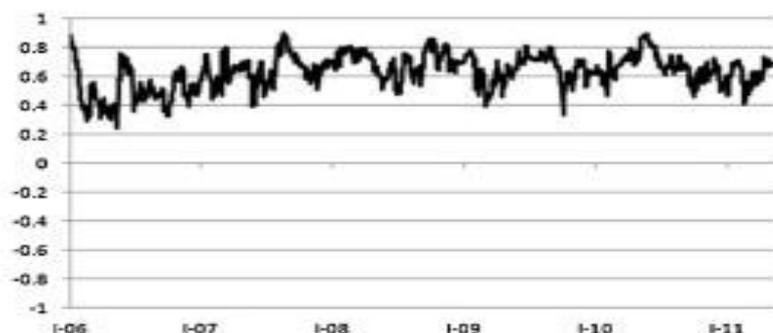
Czech Rep.



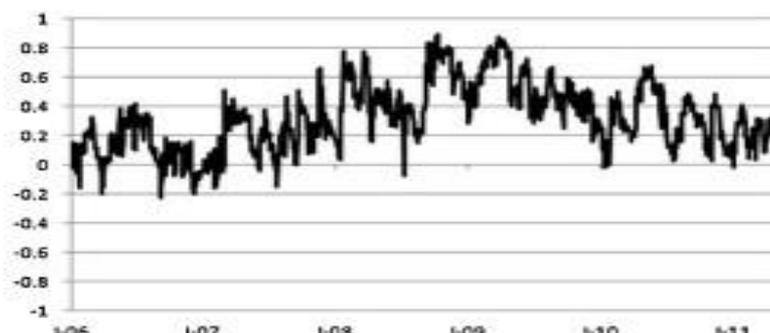
Hungary



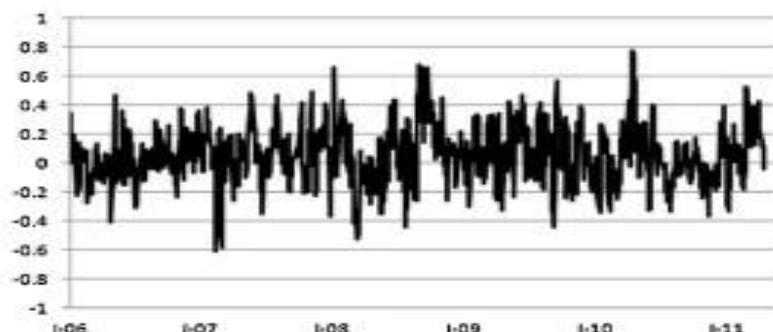
Poland



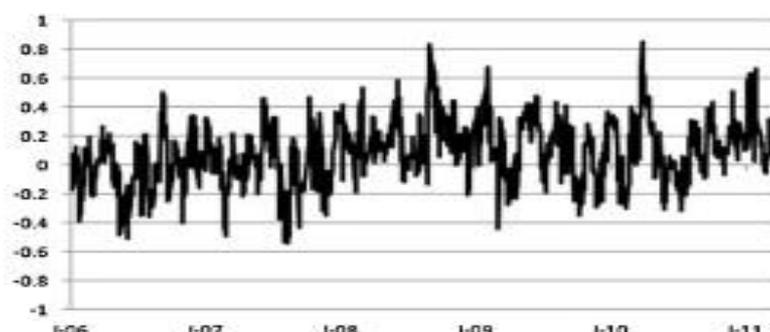
Croatia



Serbia



Macedonia



*Notes:* The figures present the conditional correlations from the estimation of a multivariate GARCH model between the returns of the STOXX index (to represent Western Europe) and the respective Central or South Eastern European stock market returns.

# Readings

- Theory:

- Engle, R. (2001): GARCH 101: The Use of ARCH/GARCH Models in Applied Econometrics, *Journal of Economic Perspectives*, pp. 157-168.
- Engle, R. (2003): Risk and Volatility: Econometric Models and Financial Practice, Nobel lecture, December 8, 2003,
- The Royal Swedish Academy of Sciences (2003): Time Series Econometrics: Cointegration and Autoregressive Conditional Heteroscedasticity

- Applications:

- Kocenda, E. (1998): Exchange Rate in Transition (chapters 2-3), CERGE-EI
- Fidrmuc, J. and R. Horvath (2008): Volatility of Exchange Rates in Selected new EU Members, *Economic Systems*, 103-118.
- Choudhry, T. (2000): Day of the week effect in emerging Asian stock markets: Evidence from the GARCH model, *Applied Financial Economics*, 235-242.

SEE THE COURSE WEBSITE FOR SOME OF THESE READINGS

- Read about multivariate GARCH in Brooks (2008) *Introductory Econometrics for Finance*

The Volatility Laboratory of R. Engle available at  
<http://vlab.stern.nyu.edu/>