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Abstract:
Understanding of volatility term structure is highly relevant both for market agents and policymakers. As traditional methodologies often bring results contradicting situation on the markets, we revisit volatility term structure modeling in univariate case. In this paper we benefit from extensive high-frequency dataset of US Treasury futures prices allowing us to empirically inspect the behaviour of the respective realized volatility term structure. We believe that the discovered properties justify the application of multi-factor modeling techniques primarily developed for yield curves. Finally we develop the comprehensive methodology fitting empirical data efficiently by term structure decomposition using Nelson-Siegel class of models.

JEL:
Keywords: Realized volatility; Term structure; Dynamic Nelson-Siegel model; High-frequency data

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1 Introduction

Existing literature on term structure modeling usually targets yield curves. However, volatility literature arrived to consensus, that implied volatility is dynamic and that Black and Scholes (1973) assumption of constant volatility across varying time to maturity is rejected by empirics. Therefore, volatility is forming a term structure, which is reflecting the fact, that uncertainty is perceived differently across varying horizons. Thorough understanding of volatility term structure is of high importance for valuation of various financial assets and its derivatives such as interest rate derivatives.

Despite the fact that volatility term structure brings additional valuable information compared to sole analysis of yield levels, publishing activity focusing on volatility term structure modeling has been rather scarce. Descriptive analyses on implied volatility term structure were compiled for instance by Stein (1989) or Xu and Taylor (1994), who arrived to conclusion that similarly to yield curves, term structure of implied volatility can take broad variety of patterns.

As Cieslak and Povala (2016) state, as long as the short end of yield term structure represents the markets expectations of monetary policy steps, volatility term structure reveals information of uncertainty associated with these steps. Therefore, ability to efficiently model volatility term structure is of interest both for policymakers and investors.

Majority of existing literature of term structure modeling works with low-dimensional affine models, where usually single factor for volatility modeling is applied. However, volatility literature is gradually converging to analogous conclusion to Litterman and Scheinkman (1991), who claimed that at least three factors are needed for proper description of interest rate terms structure. As far as volatility is concerned, much of the recent literature comes to conclusion that single-factor stochastic volatility models are not efficient and are outperformed by multi-factor volatility models in terms of capturing the volatility dynamics (e.g. Kim and Singleton (2012), Park (2011), Christoffersen et al. (2009)). Moreover, departure from single-factor volatility models has also interpretational and economic substantiation as numerous empirical studies arrived to conclusion that long-term volatility shows different behavior patterns compared to short-term volatility (Byoun et al. (2003), Heynen et al. (1994) or Mixon (2007)). Therefore, extension to multi-factor (at least two-factor) models appears logical step to match observed data.

As Derman et al. (1996), Krylova et al. (2009) or Christoffersen et al. (2009) remark, volatility term structures show analogous characteristics to yield curves. This is frequent motivation starting point for extension and application of methods primarily developed for interest rate term structure modeling also for volatilities. Similarity of behaviour of yield curve and term structure of other financial assets justifying application of modeling approaches primarily developed for interest rates has been claimed in literature by multiple authors. Among others, Hansen and Lunde (2013) or Barunik and Malinska (2016) arrived to conclusion that similarity of yield curve and oil futures term structure allow for successful application of Nelson-Siegel model for fitting respective curves on extensive dataset. Therefore, we believe potential similar properties of volatility term structure and yield curve together with the fact that existing literature continually adheres to decomposition of volatility term structure to components (often performed using principal components analysis) justifies application of Nelson-Siegel approach on volatility term structure modeling task. Moreover, we suppose that the approach and respective interpretation of the Nelson-Siegel components by Diebold and Li (2006) is able to generate good fit for volatility term structure. Diebold and Li (2006) interpreted the components as short-term, medium-term and long-term (subject to certain conditions thoroughly discussed in their paper) which suitably corresponds the above mentioned conclusion of relevant literature that volatility was detected to contain different type of information and to be differently sensitive for short, medium or long-term horizon.

As mentioned above, decomposition of yield curves to factors using various methodologies has been frequently applied both in literature and practice (see e.g. Litterman and Scheinkman (1991) or Diebold and Li (2006)). More recently, the literature has been examining also common factors in the second moment as a new source of valuable information (e.g. Jareño and Tolentino (2012), Díaz et al. (2010) or Benito and Novales (2007)). Benito and Novales (2007) tested the statistical equivalence of the volatility series estimated from a factor model for interest rates to those obtained from a factor model for volatilities. The author concluded that the results are not equal, which might be caused by loss of some information on second order moments during the estimation of yield curve factors.

Summarized, our motivation to inspect the decomposition of volatility term structures of the US Treasury futures is following. First, the realized term structure of US Treasury futures behaves differently for short and long maturities and therefore presumably contains additional valuable information relevant for various market agents. Second, proved versatility and ability of Nelson-Siegel class of models to fit wide spectrum of yield curves and term structures of various assets is likely to operate efficiently also in case of realized volatility being the efficient instrument capturing the latency of volatility. And finally, we also believe that substance of the Nelson-Siegel factors has tangible and useful interpretation with respect to the properties of the realized volatility in relation to maturity. We expect that decomposition of the whole term structure to factors with handy substantiation as long-term, short-term and medium-term components might be useful in future bond pricing research. Since we are exploring the topic with limited existing literature coverage, this paper intends to present initial inspection of the realized volatility term structure properties and formulation of "stylized facts" which will be useful in future research.

This paper is organized as follows. Section 2 presents methodology applied, focusing especially on motivation and justification of application of Nelson-Siegel approach in case of realized volatility term structure modeling, Section 3 describes data and data processing techniques, Section 4 presents the fitting procedure, Section 5 summarizes the results and Section 6 concludes.

2 Methodology

In this paper, we intend to show that it is possible to explain the universe of term structures of volatility of Treasury futures (front contracts) returns observable on the market by limited number of factors having their immediate substantiation. For this task we benefit from two key concepts recalled below - realized volatility measure and dynamic Nelson-Siegel model.

2.1 Realized volatility

Inspecting (and also modeling and forecasting) volatility is complicated by the fact that the actual volatility is not directly observable. Therefore, researchers developed multiple approaches relying on strict parametric assumptions to capture the latency of volatility. These methodologies include autoregressive conditional heteroskedasticity (ARCH) or stochastic volatility (SV) models, or alternatively, option-based implied volatility measures. As Andersen and Teräsvirta (2009) summarize, in order to approximate current and future levels of volatility, some literature also employs historical volatility measures (i.e. backward-looking sample return standard deviation), which generally do not provide with outcomes consistent with basic properties of volatility (such as mean revision).

Thanks to availability of high-frequency data on various financial assets and to increasing computational power needed for efficient processing of large-scale datasets, we can observe in recent literature stronger presence of model-free data-driven volatility measurements to the detriment of parametric conditional volatility models. As concluded by Andersen et al. (2003), simple realized volatility models show better forecasting performance compared to traditional volatility models. Following the recent surge of literature, we will employ realized volatility (RV) measures in order to accomplish the primary goal of this paper, i.e. to model volatility term structure using Nelson-Siegel approach.

First step is to construct realized volatility from high-frequency log-returns. We use *medRV* estimator as formulated by Andersen et al. (2012) constructed as:

$$medRV_t = \frac{\pi}{6 - 4\sqrt{3} + \pi} \left(\frac{N}{N-2} \right) \sum_{i=2}^{N-1} med(|r_{t,i-1}|, |r_{t,i}|, |r_{t,i+1}|)^2 \tag{1}$$

where $r_{t,i}$ generally represents the $i-$th return on trading day t and N is the number of equispaced returns on the trading day.

As discussed in Andersen et al. (2012), *medRV* performs better compared to bi-power or multi-power RV measures in terms of robustness in finite sample with respect to jumps and occurence of spurious zero returns caused by quote or trade price duplicates.

2.2 Dynamic Nelson-Siegel approach

The goal of this paper is to inspect whether there exists a possibility how to satisfactorily describe the realized volatility term structure by limited number of factors, having preferably also certain informative value. Due to its favourable properties for this task, we decided to apply dynamic version of Nelson-Siegel (DNS) as reformulated by Diebold and Li (2006). Therefore, we take advantage from the DNS model to perform cross-sectional and dynamic fit of realized volatility term structure and decompose the realized volatility term structure to long-term, medium-term and short-term factors. Our motivation to employ dynamic Nelson-Siegel model in realized volatility term structure modeling is threefold. First, as concluded among others also by Sarker et al. (2006), dynamic Nelson-Siegel proved to outperform other classes of term structure models (no-arbitrage or affine) in terms of ability to model jointly cross-sectional and time-series dimensions of yield curves while preserving also solid forecasting performance. Second, even though its functional specification is rather compact, the dynamic Nelson-Siegel model can fit wide variety of term structure shapes (increasing, decreasing or humped curves). And finally, the factors being the ultimate outcome of the model are, compared to other methods of term structure modeling, rather straightforward to interpret.

In the framework introduced by Diebold and Li (2006) for yield curves, the dynamics of the term structure Treasury futures realized volatility can be described by:

$$medRV_t(\tau) = \beta_{0t} + \beta_{1t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \beta_{2t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right) \tag{2}$$

where $medRV_t(\tau)$ is median realized volatility at time $t = 1, ..., T$ with time to maturity τ. Coefficents β_{0t}, β_{1t}, and β_{2t} are in the literature interpreted as level (being long-term component), slope (being short-term component as it decays exponentially at rate λ_t), and curvature (being medium-term component as it increases for medium term maturities and then decays for longer maturities), respectively.

As might be observed in Figure 1, loading on the level factor is equal to one and does not change with maturity. Therefore, change of β_{0t} means horizontal shift of the entire curve. Loading on the slope factor decreases from one to zero for infinite maturity (note that $medRV_t(\infty)$ - $medRV_t(0) = -\beta_1$). Finally, loading on curvature factor converges to 0 with maturity going to zero or infinity, reaching its maximum at maturity equal to $\frac{1}{\lambda}$. It is worth noting, that loading on slope factor is higher than curvature loading for shorter maturities, and therefore, affecting volatility of shorter-maturity Treasury futures relatively more. Finally, β_2 is interpreted as the curvature factor due to its zero limit loading for the maturity going to zero and infinity.

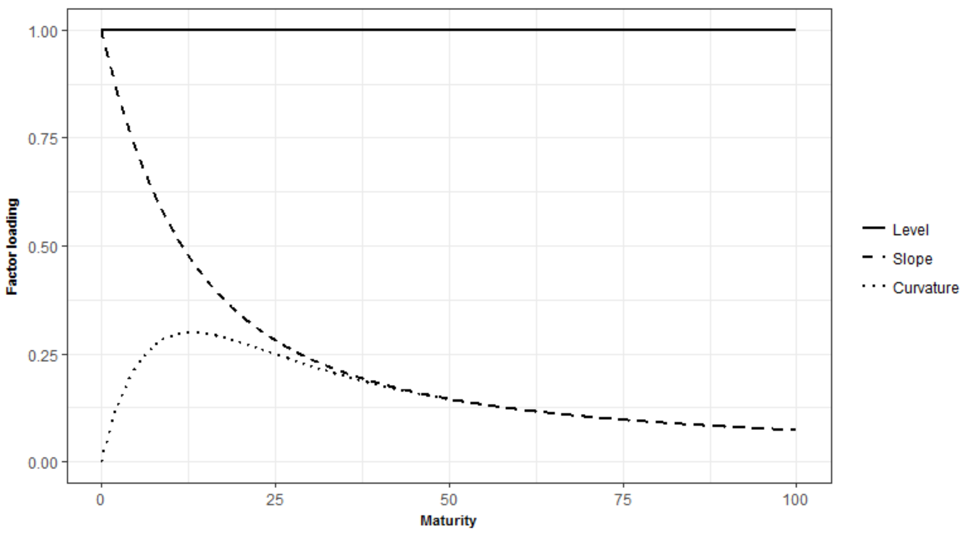


Figure 1: Nelson-Siegel factor loadings for $\lambda = 0.1379$

2.2.1 Decay parameter λ

Next to *beta* coefficients described above, fitting Nelson-Siegel model also requires to deal with the decay parameter λ. Practically, determining λ_t means solving of the trade-off between fitting the term structure for short or long maturities. There is no clear consensus in the relevant literature about any concrete value of the parameter to be used or about an approach how to derive its optimal value(s).

In principle, there are two ways how to treat λ parameter determining also whether the curve fitting task will be of linear or non-linear nature. As the ultimate task of this paper is to investigate, whether Nelson-Siegel approximation of the volatility term structure is applicable, we will empirically inspect all approaches presented below.

First, λ may be estimated together with the β coefficients, implying nonlinear (and therefore computationally much more demanding) nature of the estimation task of the Equation (2). Summarized, this option entails non-linear least squares estimation of β_{0t}, β_{1t}, β_{2t}, and λ_t for all t. Existing literature of yield curve modeling using Nelson-Siegel model often treats the non-linearity of the initial task by inclusion of grid search techniques in order to find the optimal λ_t for each period. It can be anticipated, that such an approach will probably generate very good fit to the data. However, possible instability of λ would most probably cause deterioration of predictive power of the factors as concluded by Vela (2013) in case of yield curve modeling.

Second option is to fix the value of λ, which leaves us with linear least squares estimation. In literature of dynamic Nelson-Siegel model applications, there are two most common approaches of fixing the decay parameter. First, most authors rely on argumentation provided by Diebold and Li (2006), who simply derived the value from setting medium maturity at 30 months, implying $\lambda = 0.0609$ (or $\lambda = 0.7173$ for maturity expressed in years). Alternatively, we can pursue an optimization task in order to find the optimal λ minimizing the errors of the Nelson-Siegel approximation over the whole period.

Based on the argumentation above, we believe, that fixing the decay parameter to the optimal value based on the procedure described in detail in Section 4 is the most appropriate method of treating λ as also challenged by various authors to be rather the non-transparent and abstract component of the Nelson-Siegel approximation. When discussing the results of the empirical analysis pursued in this paper, we will refer to this method being the primary for treating the decay parameter, however, the results applying the remaining two methodologies will be presented as well in order to inspect robustness of our conclusions.

To summarize, three approaches to realized volatility term structure modeling will be inspected:

Model 1: Fixed optimized λ^*

- ***Estimation:*** Fixing λ value in the DNS model to a constant equal to the optimized value λ^* being output of minimization of SSE (Equation 3) will leave us with linear problem to be solved by ordinary least squares.
- ***Output:*** Time series of three parameters $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$.

Model 2: Time-varying λ_t

- ***Estimation:*** Time-varying λ_t found by grid search allowing for linear estimation of beta coefficients.[1]
- ***Output:*** Time series of four parameters $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\lambda}$.

Model 3: Fixed λ^{DL}

- ***Estimation:*** Fixing λ value in the DNS model to a constant equal to the value used by Diebold and Li (2006) will leave us with linear problem to be solved by ordinary least squares.
- ***Output:*** Time series of three parameters $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$.

3 Data

3.1 Data description

When referring to *volatility term structure* throughout this paper, we speak about a term structure of volatility of US Treasury futures (front contract) logarithmic returns. As we are using continuous time series of front contract futures prices, the term structure is formed by maturities of the underlying assets, i.e. US government notes/bonds of 2-year, 5-year, 10-year and 30-year maturity. In order to construct a realized volatility measure, *medRV*, as described above, we use 1-minute US Treasury futures data (active contracts) from Tick Data, Inc.[2] database. We examine contracts for each US Treasury benchmark tenors, i.e. 2-year (CME global ticker: TU), 5-year (CME global ticker: FV), 10-year (CME global ticker: TY) and 30-year (CME global ticker: US).

There are multiple reasons why to analyze futures instead of cash market in order to model realized volatility term structure. First, as long as this paper applies data-driven methodology, immediate availability of clean 1-minute high-frequency futures data from renowned database is extremely beneficial. Second, observing situation on US bond market in past decade, futures market has been gaining relative importance to the cash market[3]. Third, due to delivery mechanism of US Treasury futures contracts, futures prices are tightly linked to underlying bond prices (and yields), and moreover, also due to lower transaction costs, futures market was detected to be dominant to cash market in reaction to news and price discovery process (see e.g. Brandt et al. (2007), Andersen et al. (2007) or Engle (1998)). Panzarino et al. (2016) found that volatility on the futures market tends to spread to cash market, whereas the reverse flow is rather much weaker.

For our analysis we restrict ourselves to futures price observations in the period from 1/2/2001 to 12/31/2015. The selection of the inspected timeframe is beneficial, as long as throughout the entire period the futures contracts have consistent specifics and delivery conditions especially in terms of annual coupon rate of the underlying bond contract which changed to 6% in 2000. We believe that 15 years of 1-minute high-frequency observations covering

[1]We benefit from Nelson.Siegel function in R included in *YieldCurve* package by Sergio Guirreri (`https://cran.r-project.org/web/packages/YieldCurve/YieldCurve.pdf`).

[2]http://www.tickdata.com/

[3]See The New Treasury Market Paradigm, CME Group, June 2016, available at https://www.cmegroup.com/education/files/new-treasury-market-paradigm.pdf.

also turbulent period of recent financial crisis represent wide variety possible situations on the respective markets reflected in various shapes of volatility term structures.

Figure 2 presents time series of US Treasury futures prices in the period from January 2001 to December 2015.

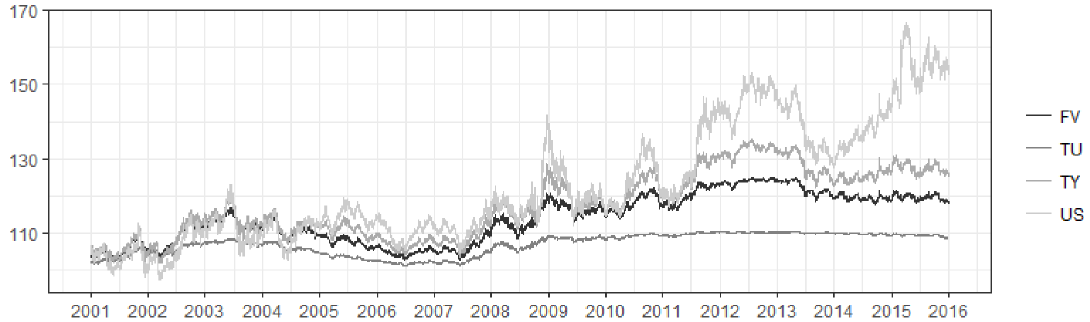


Figure 2: US Treasury futures prices development

The data show diverging futures prices, especially in case of the longest 30-year tenor. As the bond prices are inversely related to yield levels, the significant growth price of the 30-year bond future relatively to the shorter tenors reflect the flattening of the US yield curve (measured by decreasing spread between 30-year and 2-year bonds) observable on the market since the global financial crisis. General rise in the Treasury futures prices due to low-interest policy pursued by the Federal Reserve (and associated uncertainty and speculations of the potential policy change) impacted the long-term contracts more due to their inherent higher sensitivity to the interest rates changes.

3.2 Data processing

The raw high-frequency data on US Treasury futures prices are clean and validated by TickData in-house system. However, we need to perform several more steps in order to acquire solid and representative time series for meaningful calculation of realized volatility.

First step is to exclude non-active days such as weekends or public holidays in the USA. We also drop days having only a single unique futures price observation during the trading day. This procedure leaves us with 3,814 days.

In order to inspect trading activity on US Treasury futures market we plot intra-day distribution of trading volume by calculating mean over the entire period of volumes traded in a given minute (see Figure 3). We observe that largest activity is present during Chicago Mercantile Exchange trading hours, i.e. 07:20 to 14:00 CT. However, due to operation of trading electronic platform CME Globex, significant trading activity is observable also outside the CME trading hours. Therefore, we decide to extend the interval for purposes of realized volatility calculation by two hours on each pole and to define the trading day for our realized volatility calculation purposes from 05:20 CT to 16:00 CT in order to include all significant activity to our calculations (see the shaded area in Figure 3). Moreover, this window includes the regular announcements issued by Federal Reserve System and other relevant authorities which represent significant determinants of changes on the US Treasury market (Andersen and Benzoni, 2010).

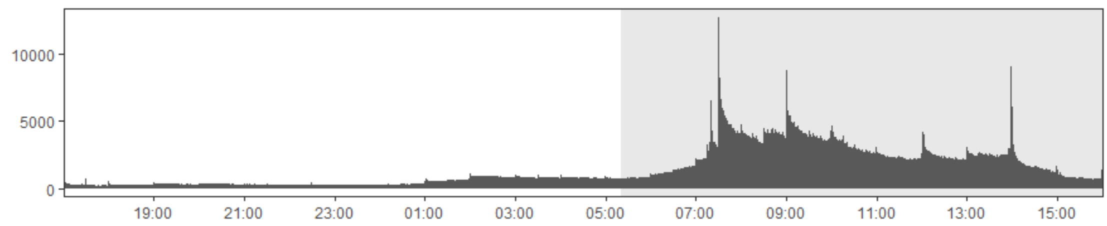


Figure 3: Trading activity: sum of mean trading volumes for TU, FV, TY and US

Based on the findings in the relevant literature (summarized in e.g. Liu et al. (2015), Hansen and Lunde (2006)), we aggregate our data to 5-minutes sampling interval in order to benefit from optimal trade-off between bias and variance, which leaves us with final number of 491,981 observations.

4 Fitting the realized volatility term structure

Having the data processed using the procedure described in the preceding section, the fitting starts with calculation of log-returns which are presented together with their descriptive statistics presented in the Appendix (Figure 12 and Table 1).

As presented in the Section 2.1., we use *medRV* estimator introduced by Andersen et al. (2012). Figure 4 sets forth the resulting time series of realized volatility of US Treasury futures log returns for four tenors - 2-year, 5-year, 10-year and 30-year. Before moving to the fitting procedure, we consider highly beneficial to discuss the discovered properties of realized volatility term structures estimated over the sample as to our knowledge the number of works empirically examining the properties of the US Treasury (futures) realized volatility in relation to time to maturity is very limited.

Generally, volatility of bond futures prices depend on the volatility of the respective yield to maturity of the underlying asset and on their duration. Therefore, in case that the yield volatility is stable across maturities, then the futures bond price volatility will be upward sloping because of the increasing duration. Consistently with this key market principle, both in terms of the mean and median during the inspected period, realized volatility is higher for longer maturities, which implies the term structure of the realized volatility to be on average upward sloping. Also variability (in terms of standard deviation) of realized volatility in case of longer maturities is relatively higher compared to shorter maturities. Moreover, realized volatility for all the maturities showed to be highly persistent (with relatively higher persistence in case of short end of the term structure). Significant differences of the behaviour of short-end and long-end of the realized volatility term structure support the suitability of multi-factor modeling

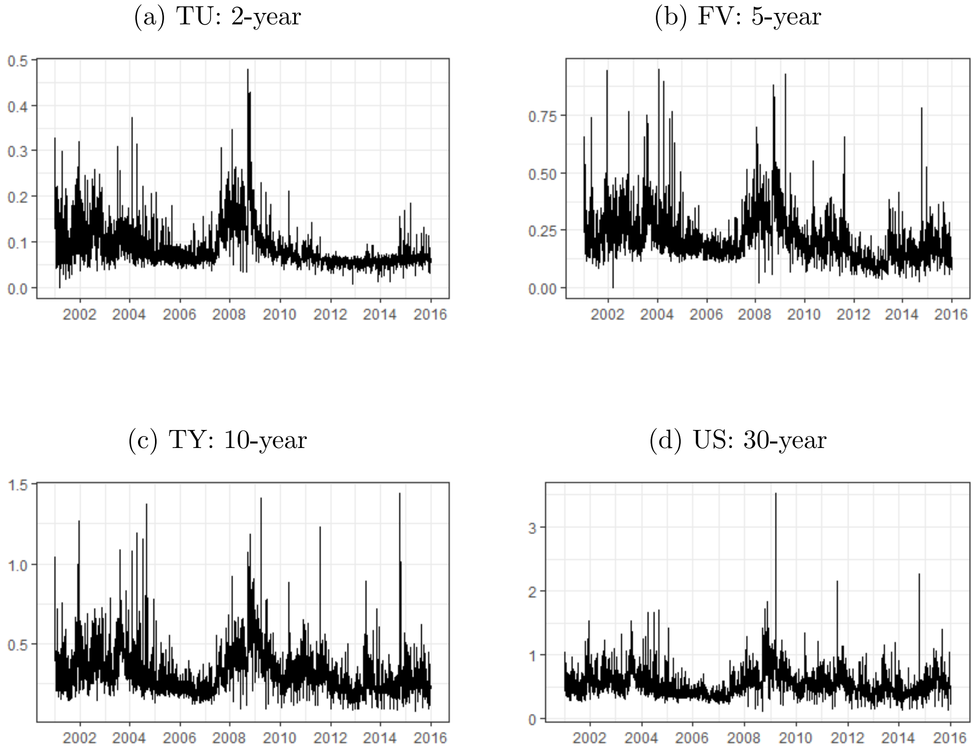


Figure 4: Realized volatility of Treasury futures log-returns

techniques.

The fact that the mean realized volatility term structure is (at least for our sample) upward-sloping is illustrated on Figure 5 showing the mean term structure as implied by observations on 2-year, 5-year, 10-year and 30-year maturity along with the respective standard deviation band. It appears that a basic statistical analysis of the rich sample of 3,814 term structures confirms the the intuition that longer time to maturity is associated with higher risk measured by realized volatility, and that the volatility term structure contains potentially valuable additional information as it probably exhibits fundamentally different properties at its short and long-end. Further details on descriptive statistics of the realized volatility series are summarized in the Appendix (Table 7 and Figure 13).

4.1 Model 1: Fixed optimized λ^*

As presented in the Section 2, one of the options how to treat the decay parameter λ is to fix the coefficient to a constant value based on a criteria representing the quality of the fit. In this respect, we decided to find the optimal λ by minimizing the sum of square errors:

$$\lambda^* = \arg\min_{\lambda \in \Theta} \sum_{t=1}^{T} \sum_{i=1}^{4} (RV_t(\tau_i) - \hat{RV}_t(\tau_i; \beta_{0t}, \beta_{1t}, \beta_{2t}, \lambda))^2 \tag{3}$$

Solving Equation 3 provides us with the solution for optimal lambda $\lambda^* = 0.1379$. At the

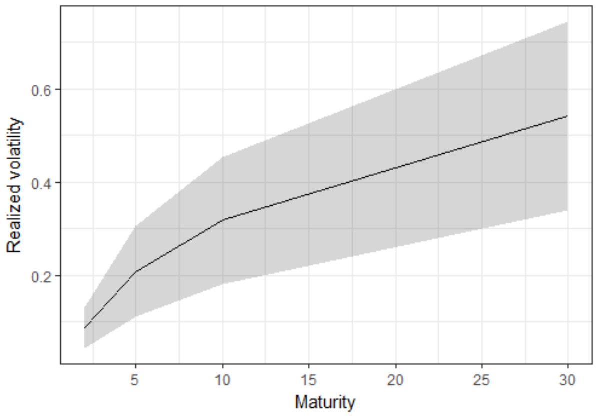


Figure 5: Mean realized volatility terms structure of US Treasury futures

first sight, calculated λ^* differs from the one used by Diebold and Li (2006), who fixed the value at 0.7173 (for maturity in years). Let us inspect, at what maturity the calculated λ^* maximized the loading on the curvature factor from Equation 2. Maximum loading of the curvature factor for $\lambda = \lambda^*$ (C_{MAX}) equals 0.2984 (see Figure 1), therefore the respective medium-term maturity shall be the solution to the following:

$$\frac{1 - e^{-\lambda^* \tau_{medium}}}{\lambda^* \tau_{medium}} - e^{-\lambda^* \tau_{medium}} = C_{MAX} \tag{4}$$

The single satisfactory (i.e. non-negative) solution of Equation 4 determines the medium-term time to maturity to be approx. 7.5 years. As often pronounced by practitioners[4], usually 5 to 10 years is considered as medium-term horizon in case of bond markets, which fully corresponds to our result.

Fitting of *Model 1* is expressed by the following equation:

$$medRV_t(\tau) = \beta_{0t} + \beta_{1t} \left(\frac{1 - e^{-\lambda^* \tau}}{\lambda^* \tau} \right) + \beta_{2t} \left(\frac{1 - e^{-\lambda^* \tau}}{\lambda^* \tau} - e^{-\lambda^* \tau} \right) \tag{5}$$

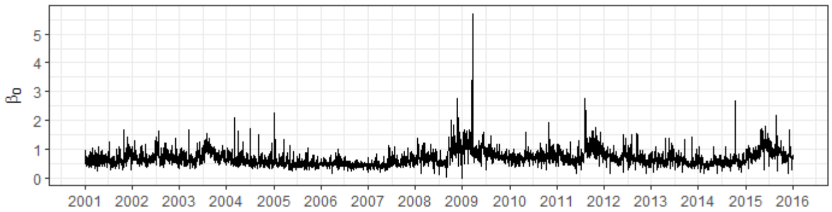
Results of the ordinary least squares estimation of the Equation 5 are shown in the Figure 6. Similarly to conclusions presented by Guo et al. (2014) for implied volatility estimation, we observe also in case of realized volatility increased instability of the factors during the period of the financial crisis (September 2008 - December 2009).

Level factor $\hat{\beta}_0$ is always positive throughout the period and is nearly perfectly negatively correlated with the slope factor $\hat{\beta}_1$. Moreover, as presented in Figure 1 in Section 2, loading on the slope factor decreases to zero with maturity going to infinity whereas the loading on the level factor remains constant, which means that for long maturities the fitted realized volatility corresponds to the value of the level factor. On the contrary, for short maturities, due to the nearly perfect negative correlation between $\hat{\beta}_0$ and $\hat{\beta}_1$, the relative importance of the level factor is more limited as it becomes to a large extent offset by the slope factor. Moreover, level factor is the most stable of the three factors supporting the idea that level factor resembles the long-term volatility component.

[4]See for example https://www.investopedia.com/terms/m/mediumterm.asp

In order to inspect stationarity of the estimated factors we performed the augmented Dickey-Fuller (ADF) test. In case of all three coefficients, the ADF test strongly rejects the null hypothesis of unit root presence in the series. Results of the test together with summary of descriptive statistics of the three estimated coefficients are presented in Appendix (Table 2).

(a) Level



(b) Slope

(c) Curvature

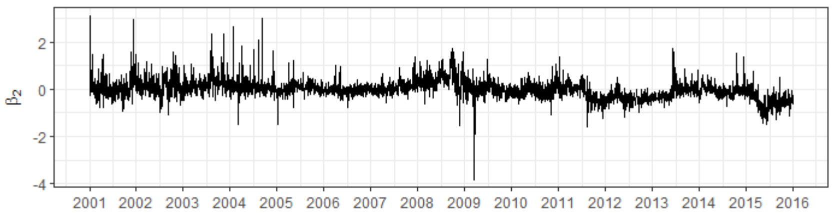
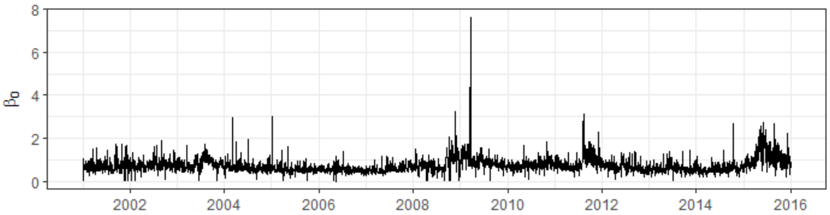


Figure 6: Model 1: Estimated β coefficients with fixed optimized $\lambda^* = 0.1379$ for 2001 - 2015

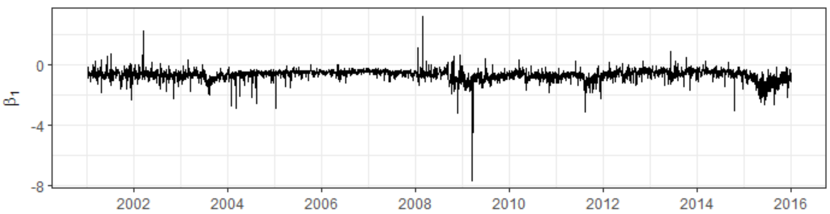
4.2 Model 2: Time-varying λ_t

Allowing for time-varying λ_t coefficient requires either non-linear least square for estimation of the model formulated in the Equation 2 or optimizing an optimal value of λ_t for each period prior linear estimation. In order to avoid the complexity of the non-linear estimation we proceed with a grid search for optimal λ_t. As already mentioned, the estimation results in four fitted time series, which are presented in Figure 7. Consistently with the Model 1, the $\hat{\beta}_0$ coefficient remains positive throughout the sample period. However, in case of time-varying lambda, the $\hat{\beta}_0$ coefficient is more persistent and more stable than $\hat{\beta}_1$ and $\hat{\beta}_2$. Correlation of the coefficients show the similar patterns as in the previous case, most importantly, the $\hat{\beta}_0$ and $\hat{\beta}_1$ are nearly perfectly negatively correlated. Such similarity in properties of the estimated factors might be assigned to the fact that median value of λ_t equals 0.156 and is rather close to the optimized λ^* (0.138). Also in this case the ADF test rejected the null hypothesis of unit root presence. Analogically with the previous model, the coefficients report lower volatility in the pre-crisis period (prior September 2008).

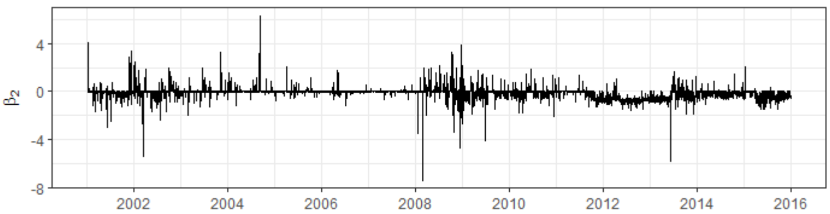
(a) Level



(b) Slope



(c) Curvature



(d) Lambda

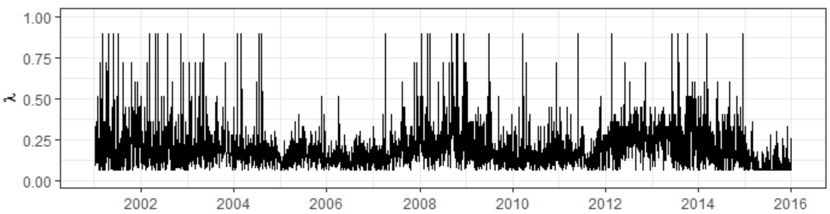
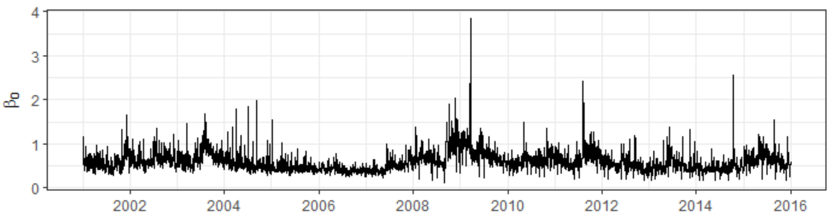


Figure 7: Model 2: Estimated β coefficients with time-varying λ_t for 2001 - 2015

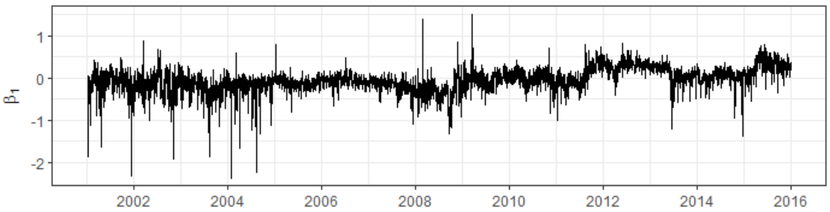
4.3 Model 3: Fixed λ^{DL}

Model 3 relies on the adoption of the value of the decay parameter as determined by Diebold and Li (2006) at $\lambda^{DL} = 0.7173$ (for maturity expressed in years). Significant difference in the value of the decay parameter λ^{DL} compared to decay parameters applied (or estimated) in the former models is reflected in selected statistical properties of the estimated factors. First, the level factor consistently with the former models remains positive and the most stable of the three factors. The most notable difference is reported in correlation of $\hat{\beta}_0$ and $\hat{\beta}_1$ where the absolute value of the correlation coefficient is significantly lower than in the previous two cases. ADF tests against reject the null hypothesis of unit root in all series.

(a) Level



(b) Slope



(c) Curvature

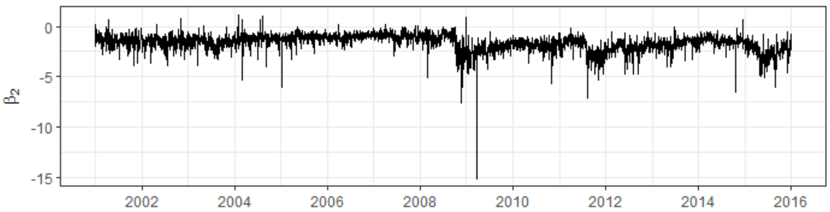
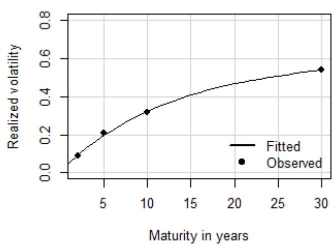


Figure 8: Model 3: Estimated β coefficients with fixed $\lambda^{DL} = 0.7173$ for 2001 - 2015

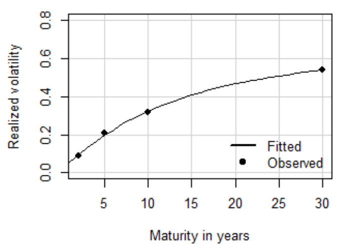
5 Results

The Nelson-Siegel modeling approaches used by the literature for yield curves as presented above differ in quality of the fit. Figure 9 shows the fit generated by each of the models of the average realized volatility term structure.

(a) Model 1



(b) Model 2



(c) Model 3

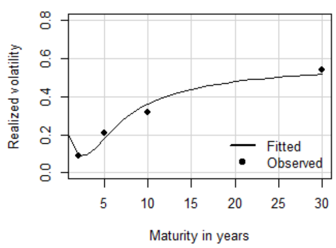
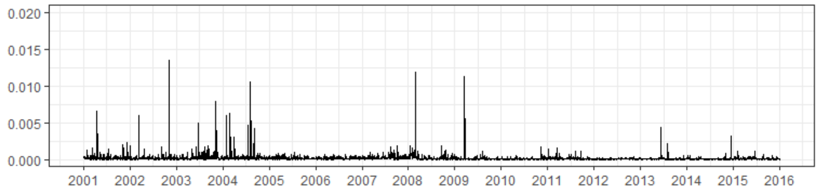


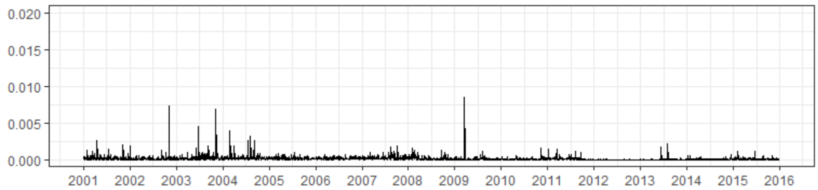
Figure 9: Average term structure fit

From mere visual assessment is obvious that Model 3 (adopting the fixed decay parameter as determined by Diebold and Li (2006)) generates the worst fit, whereas the fit of the Model 1 (fixed optimized decay parameter) and Model 2 (time-varying decay parameter) succeeded to approximate the curve more accurately.

(a) Model 1



(b) Model 2



(c) Model 3

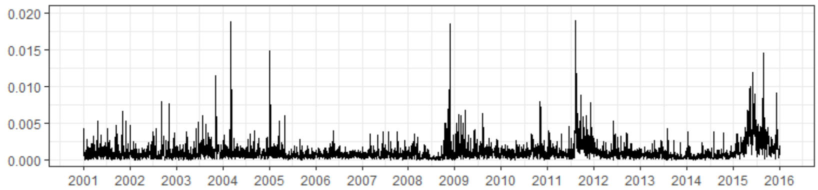


Figure 10: Mean square error of realized volatility term structure fit for 2001 - 2015

Figure 10 plots time series of mean square errors of the term structure fit for each model. In terms of MSE of all maturities, Model 3 has been clearly outperformed by the other two models. As expected, the estimation allowing for time-varying decay parameter (Model 2) fitted the realized volatility term structure the most accurately.

Both in case of Model 1 (relying on the optimized value of the decay parameter) and Model 2 (allowing for time-varying decay parameter), the models are especially successful in modeling the closest (2-year) and the furthest (30-year) tails of the term structure.

When assessing the MAE, MSE and RMSE of the entire term structure fit, Model 2 (time-varying λ_t) reports the lowest errors, closely followed by Model 1 (fixed optimal λ^*). Model 3 outperforms the competing models only in case of the 2-year maturity. In the medium horizon, Model 2 reports the lowest errors across the measures. In case of the farthest end of the realized volatility term structure (30-year), the conclusion is not unambiguous as according to MSE (and RMSE) Model 1 reports the lowest errors[5]. Recalling the properties of the decay parameter λ, it is worth noting that lower decay parameter is reflected in better fit of the curve at long maturities, and vice versa, large λ favors the fit at short maturities.

As stated earlier, dynamic Nelson-Siegel model is capable to fit a wide spectrum of the curve shapes with high degree of accuracy (Diebold and Li, 2006). Figure 11 sets forth three examples of the realized volatility term structure observed on the respective date together with the curve fitted by Model 1 (fixed optimal λ^*). Figure 11a shows the term structure on August 22, 2013, where the Model 1 fit reported the lowest MSE (6.7e$-$12) in the inspected period.

[5] According to MAE, Model 2 reports the lowest error.

On the contrary, the highest MSE (0.01349) was reported in case of the realized volatility term structure observed on November 6, 2002 which is presented in Figure 11b. In line with the literature inspecting application of DNS model for various asset classes (e.g. Diebold and Li (2006), Hansen and Lunde (2013)), we confirm that DNS model exhibits lower accuracy in case of dispersed observations with interior extremes. Ability of Model 1 to fit also humped curves is presented in Figure 11c which captures the term structure as of December 24, 2008.

(a) August 22, 2013 (b) November 6, 2002 (c) December 24, 2008

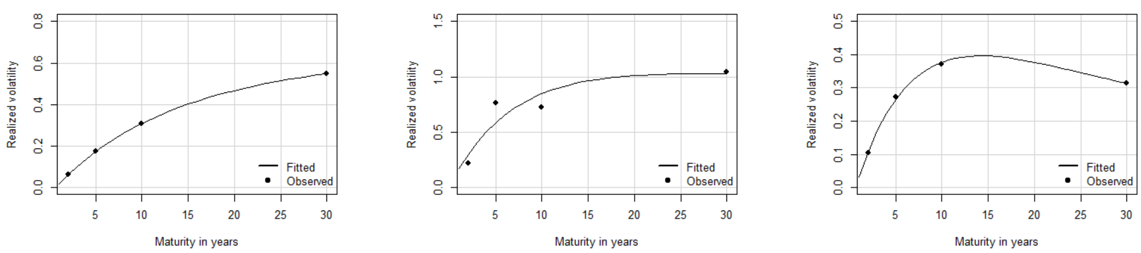


Figure 11: Fitted vs. observed volatility term structure
Model 1: Fixed optimal λ_t

6 Conclusion

In this paper, we have extended the existing scarce literature inspecting the properties of realized volatility term structure of US government bond (futures) prices. Benefiting from large sample of high-frequency data, we have found that the term structure is on average clearly upward sloping which corresponds to the general principles of volatility-maturity relationship on the bond market.

To our knowledge, this paper is pioneering the decomposition of the realized volatility term structure using dynamic Nelson-Siegel model as formulated by Diebold and Li (2006). In order to perform the initial technical inspection of this topic, we have examined three modifications of the dynamic Nelson-Siegel model and we conclude that the Model 1 with fixed optimal decay parameter λ^* provides the best balance of high accuracy, straightforward interpretation and promising potential for forecasting tasks.

In the first place, the model resulting in a simple linear representation of the term structure proved to be able to fit the extensive sample of realized volatility term structure shapes with high accuracy. Even though it was generally outperformed by the model allowing for time-varying λ_t, we believe the other benefits of the model to fully justify the preference. It is worth noting, that one of the main advantages of the general Nelson-Siegel decomposition is the dimension reduction of the term structure. We have demonstrated that the realized volatility term structure can be precisely described by three factors ($\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$). Further, fixing the decay parameter prior the series of cross-sectional regressions allows for consistent interpretation of the factors as level (being the long-term component), slope (being the short-term component), and curvature (being the medium-term component). Moreover, the fixed optimal value of λ^*

turned out to be in perfect line with the market conditions on the bond market implying 7.5 years to represent the medium time to maturity. Keeping in mind the obvious next stage of the presented research, i.e. forecasting, the fixed decay parameter prevents the deterioration of the predictive power of the individual factors as often argued in the literature.

We have shown that the volatility term structure can be accurately decomposed to factors with versatile interpretation (being short-, long-, medium-term) with promising potential to contain valuable information for exploring the risk-return tradeoff in this asset class.

Apart from that, the immediate extension consists in exploring the quality of forecasts pursuant the Nelson-Siegel decomposition. Inspired by Diebold et al. (2008), the fact that the determined factors have concrete interpretation and specific properties opens opportunity also for future research investigating links between these factors of volatility term structure of government bonds (or bills) in the world.

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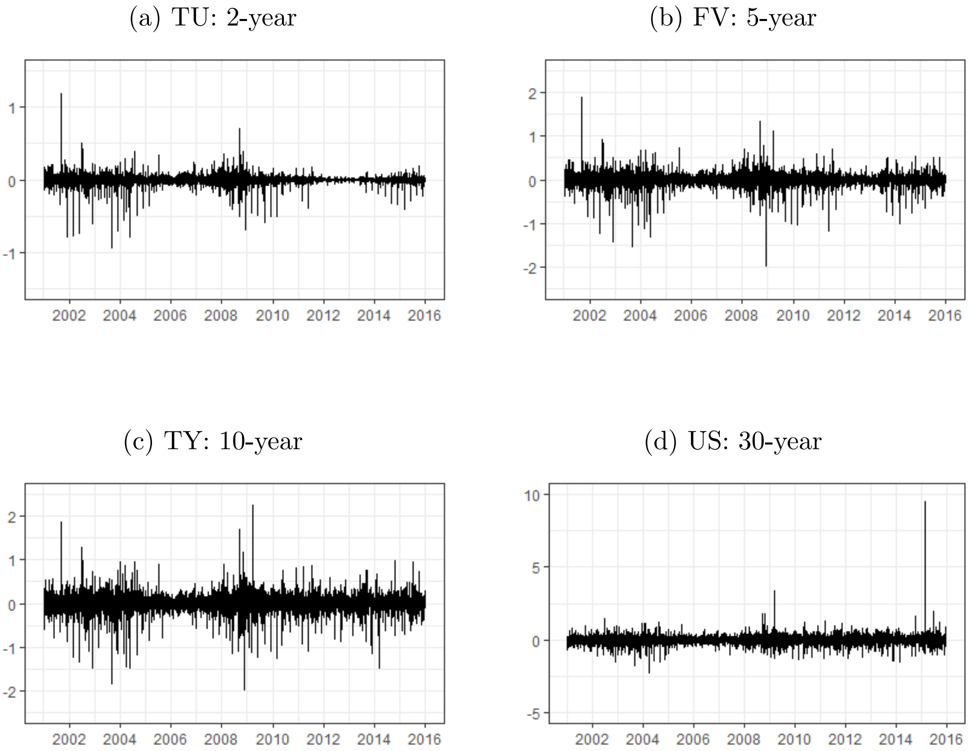
Appendix



(a) TU: 2-year

(b) FV: 5-year

(c) TY: 10-year

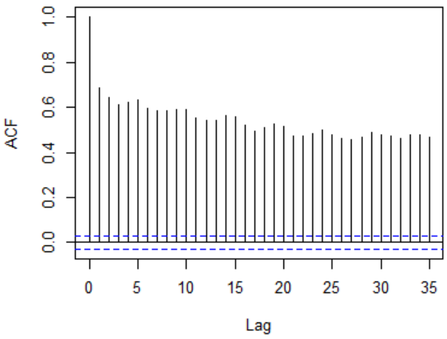
(d) US: 30-year

Figure 12: Treasury futures log-returns in percent

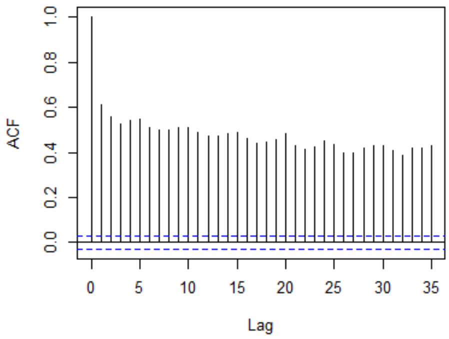
I. Basic statistics				
	TU	FV	TY	US
Maturity	2 years	5 years	10 years	30 years
N	491,981	491,981	491,981	491,981
Min	-0.937	-1.956	-1.974	-2.210
Median	0.000	0.000	0.000	0.000
Mean	0.000	0.000	0.000	0.000
Max	1.187	1.896	2.251	9.492
Standard deviation	0.011	0.026	0.037	0.060
II. Correlations				
	TU	FV	TY	US
TU	1.000			
FV	0.718	1.000		
TY	0.628	0.883	1.000	
US	0.481	0.728	0.824	1.000
III. Augmented Dickey-Fuller test				
	TU	FV	TY	US
Test statistic	-80.188	-78.664	-77.989	-77.832
P-value	<0.01	<0.01	<0.01	<0.01

Table 1: Descriptive statistics of US Treasury futures log-returns (expressed in percent)

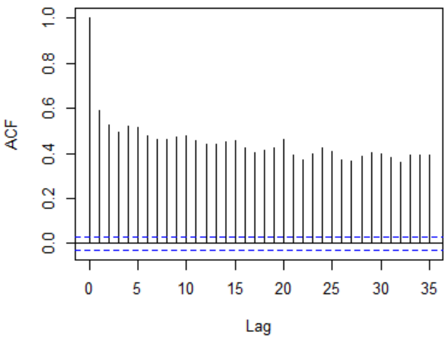
(a) TU: 2-year



(b) FV: 5-year



(c) TY: 10-year



(d) US: 30-year

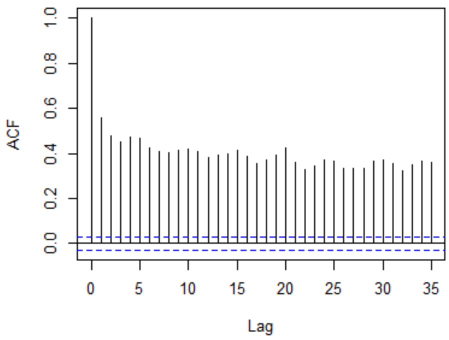


Figure 13: Autocorrelation of daily realized volatility

I. Basic statistics	β_0	β_1	β_2
N	3,814	3,814	3,814
Min	0.036	-5.680	-3.801
Median	0.659	-0.650	-0.025
Mean	0.710	-0.705	-0.011
Max	5.706	-0.087	3.115
Standard deviation	0.277	0.287	0.410
$\rho(10)$	0.390	0.410	0.418
$\rho(50)$	0.284	0.307	0.304
$\rho(100)$	0.181	0.193	0.234

II. Correlations	β_0	β_1	β_2
β_0	1.000		
β_1	-0.988	1.000	
β_2	-0.277	0.207	1.000

III. Augmented Dickey-Fuller test	β_0	β_1	β_2
Test statistic	-6.075	-5.889	-6.762
p-value	< 0.01	< 0.01	<0.01

Table 2: Descriptive statistics of the estimated factors
Model 1: fixed optimal λ^*

I. Basic statistics	β_0	β_1	β_2	λ
N	3,814	3,814	3,814	3,814
Min	0.001	-7.680	-7.387	0.060
Median	0.657	-0.638	-0.004	0.156
Mean	0.728	-0.708	-0.139	0.190
Max	7.648	3.209	6.357	0.897
Standard deviation	0.340	0.376	0.516	0.126
$\rho(10)$	0.373	0.327	0.168	0.150
$\rho(50)$	0.268	0.238	0.143	0.117
$\rho(100)$	0.168	0.162	0.092	0.097
II. Correlations	β_0	β_1	β_2	λ
β_0	1.000			
β_1	-0.924	1.000		
β_2	-0.218	-0.040	1.000	
λ	-0.212	0.312	-0.475	1.000
III. Augmented Dickey-Fuller test	β_0	β_1	β_2	λ
Test statistic	-6.271	-6.751	-8.707	-8.274
p-value	<0.01	<0.01	<0.01	<0.01

Table 3: Descriptive statistics of the estimated factors
Model 2: time-varying λ_t

I. Basic statistics	β_0	β_1	β_2
N	3,814	3,814	3,814
Min	0.134	-2.355	-15.011
Median	0.550	-0.042	-1.513
Mean	0.596	-0.043	-1.648
Max	3.856	1.516	1.181
Standard deviation	0.225	0.292	0.845
Rho(10)	0.426	0.470	0.452
Rho(50)	0.308	0.361	0.361
Rho(100)	0.220	0.290	0.236

II. Correlations	β_0	β_1	β_2
Beta 0	1.000		
Beta 1	-0.154	1.000	
Beta 2	-0.709	-0.576	1.000

III. Augmented Dickey-Fuller test	Beta 0	Beta 1	Beta 2
Test statistic	-5.845	-6.319	-5.735
P-value	<0.01	<0.01	<0.01

Table 4: Descriptive statistics of the estimated factors
Model 3: Fixed λ^{DL}

I Mean Absolute Error	All	TU	FV	TY	US
Model 1	0.008043	0.006400	0.014489	0.009686	0.001597
Model 2	**0.006161**	0.004566	**0.010729**	**0.007755**	**0.001593**
Model 3	0.023989	**0.002709**	0.024552	0.045270	0.023427
II. Mean Square Error	All	TU	FV	TY	US
Model 1	0.000159	0.000075	0.000385	0.000172	**0.000005**
Model 2	**0.000114**	0.000047	**0.000254**	**0.000145**	0.000011
Model 3	0.001066	**0.000010**	0.000801	0.002724	0.000729
III. Root Mean Square Error	All	TU	FV	TY	US
Model 1	0.012617	0.008666	0.019620	0.013116	**0.002163**
Model 2	**0.010697**	0.006883	**0.015928**	**0.012048**	0.003384
Model 3	0.032649	**0.003123**	0.028303	0.052188	0.027007

Table 5: Selected goodness-of-fit measures

Model 1	TU	FV	TY	US
Min	-0.080	-0.169	-0.121	-0.019
Median	-0.004	0.009	-0.006	0.001
Mean	-0.004	0.010	-0.006	0.001
Max	0.075	0.181	0.113	0.020
Standard deviation	0.008	0.017	0.011	0.002
Model 2	TU	FV	TY	US
Min	-0.075	-0.130	-0.138	-0.036
Median	-0.003	0.007	-0.005	0.001
Mean	-0.004	0.010	-0.007	0.001
Max	0.061	0.147	0.080	0.076
Standard deviation	0.005	0.012	0.010	0.003
Model 3	TU	FV	TY	US
Min	-0.035	-0.092	-0.580	-0.088
Median	-0.002	0.023	-0.042	0.022
Mean	-0.003	0.024	-0.044	0.023
Max	0.010	0.314	0.170	0.300
Standard deviation	0.002	0.015	0.027	0.014

Table 6: Descriptive statistics of residuals

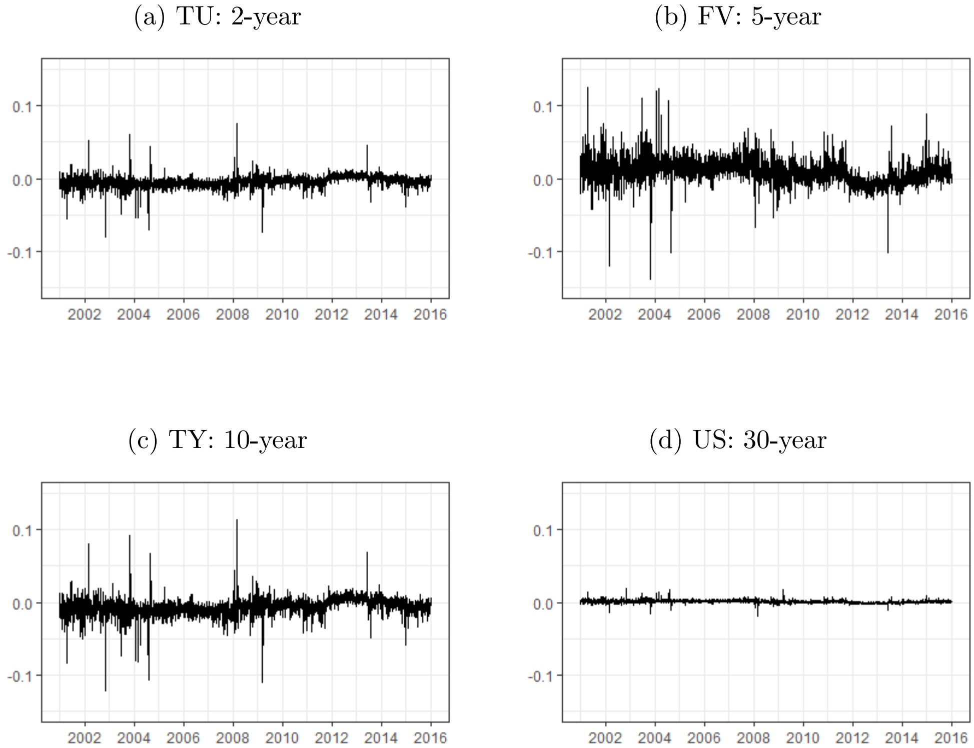


Figure 14: Time series of residuals for 2001 - 2015
Model 1: fixed optimal λ^*

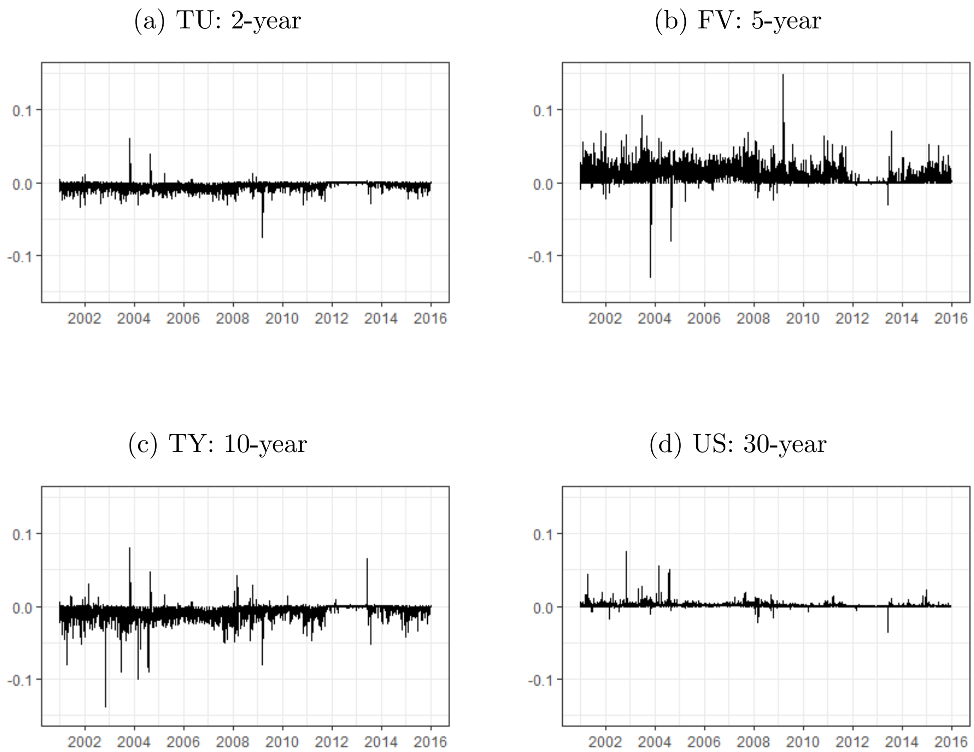
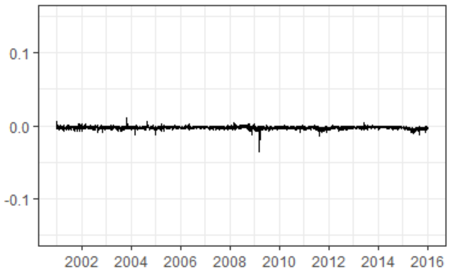
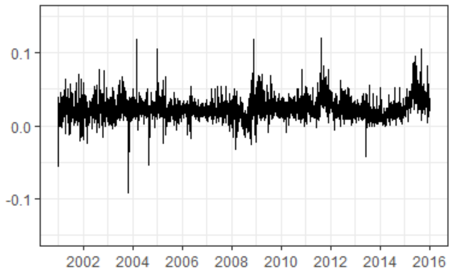


Figure 15: Time series of residuals for 2001 - 2015
Model 2: time-varying λ_t

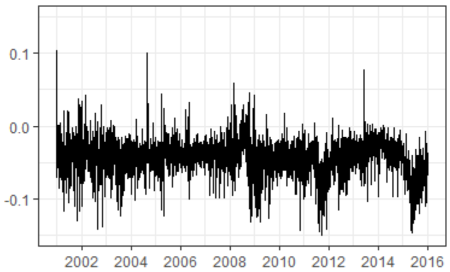
(a) TU: 2-year



(b) FV: 5-year



(c) TY: 10-year



(d) US: 30-year

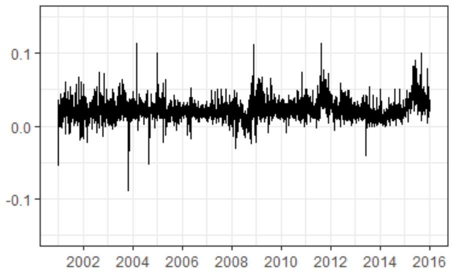


Figure 16: Time series of residuals for 2001 - 2015
Model 3: Fixed λ^{DL}

I. Basic statistics	TU	FV	TY	US
Maturity	2 years	5 years	10 years	30 years
N	3,814	3,814	3,814	3,814
Min	0.000	0.000	0.079	0.114
Median	0.074	0.186	0.285	0.499
Mean	0.088	0.208	0.318	0.541
Max	0.478	0.948	1.440	3.529
Standard deviation	0.044	0.098	0.137	0.203
$\rho(10)$	0.590	0.512	0.478	0.422
$\rho(50)$	0.456	0.383	0.348	0.302
$\rho(100)$	0.378	0.326	0.280	0.218
II. Correlations	TU	FV	TY	US
TU	1.000			
FV	0.862	1.000		
TY	0.767	0.934	1.000	
US	0.588	0.791	0.904	1.000
III. Augmented Dickey-Fuller test	TU	FV	TY	US
Test statistic	-5.671	-5.924	-5.813	-5.916
p-value	<0.01	<0.01	<0.01	<0.01

Table 7: Descriptive statistics of US Treasury futures realized volatility for 2001 - 2015

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