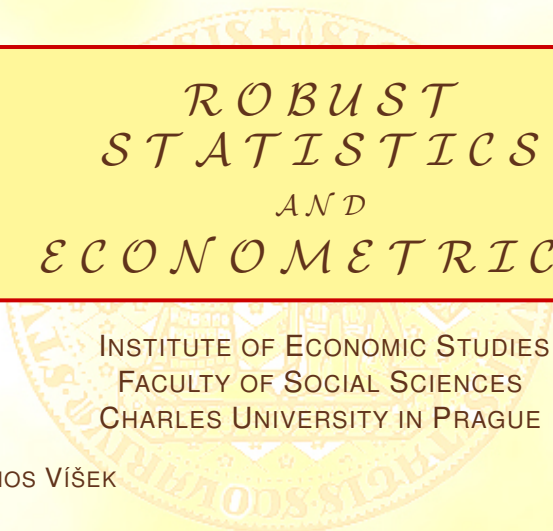




INSTITUTE OF ECONOMIC STUDIES, FACULTY OF SOCIAL SCIENCES
CHARLES UNIVERSITY IN PRAGUE (*established 1348*)



*ROBUST
STATISTICS
AND
ECONOMETRICS*

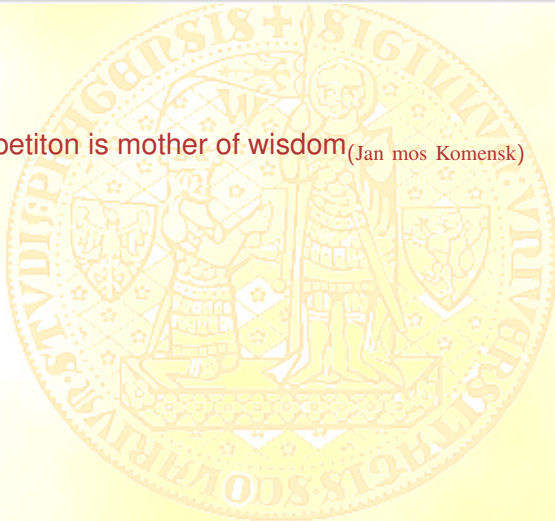
INSTITUTE OF ECONOMIC STUDIES
FACULTY OF SOCIAL SCIENCES
CHARLES UNIVERSITY IN PRAGUE

JAN ÁMOS VÍŠEK

Week 4

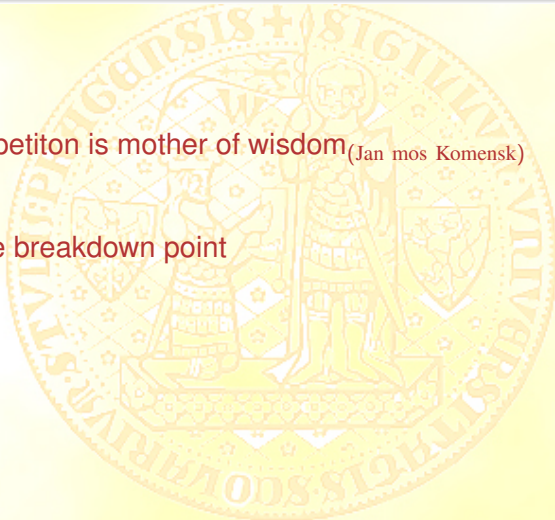
Content of lecture

- 1 Repetition is mother of wisdom (Jan mos Komensk)



Content of lecture

- 1 Repetition is mother of wisdom (Jan mos Komensk)
- 2 The breakdown point



Content of lecture

- 1 Repetition is mother of wisdom (Jan mos Komensk)
- 2 The breakdown point
- 3 Specification of robustness characteristics for classical estimators

Content

- 1 Repetition is mother of wisdom (Jan mos Komensk)
- 2 The breakdown point
- 3 Specification of robustness characteristics for classical estimators

A brief repetition of some points from previous lectures

We have concluded:

The requirements overtaken from the classical statistics

- 1 Consistency (typically weak, i. e. in probability)
- 2 \sqrt{n} -consistency (root-n-consistency)
- 3 Asymptotic normality
- 4 Loss of efficiency as small as possible
- 5 Scale- and regression-equivariance

A brief repetition of some points from previous lectures

Then we added:

The requirements enlarging the classical paradigm

- 1 Gross-error sensitivity

$$\gamma^* = \sup_{x \in R} |IF(x, T, F)|$$

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$$\lambda^* = \sup_{x, y \in R} \left| \frac{IF(x, T, F) - IF(y, T, F)}{x - y} \right|$$

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- 4 Breakdown point - will be discussed as the first topic today

A brief repetition of some points from previous lectures

Then we added *(continued)*:

The requirements enlarging the classical paradigma

- 1 Tight algorithm and reliable implementation
- invention and verification

A brief repetition of some points from previous lectures

Then we added (*continued*):

The requirements enlarging the classical paradigma

- 1 Tight algorithm and reliable implementation
- invention and verification
- 2 Good heuristic - to convince people to employ it

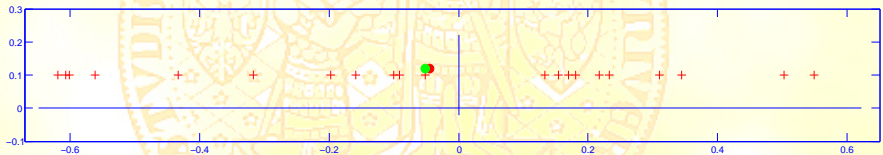
Content

- 1 Repetition is mother of wisdom (Jan mos Komensk)
- 2 The breakdown point
- 3 Specification of robustness characteristics for classical estimators

First of all some inspiration

Observe the mean ● and the median ●

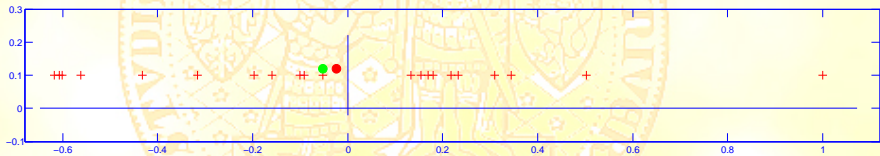
Non-contaminated data



First of all some inspiration

Observe the mean ● and the median ●

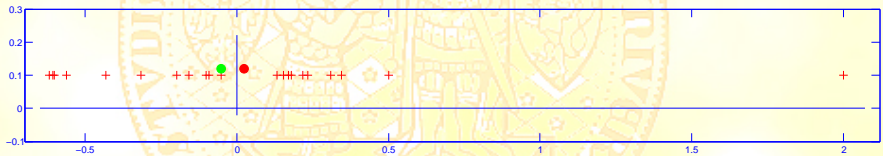
Contamination at point 1



First of all some inspiration

Observe the mean ● and the median ●

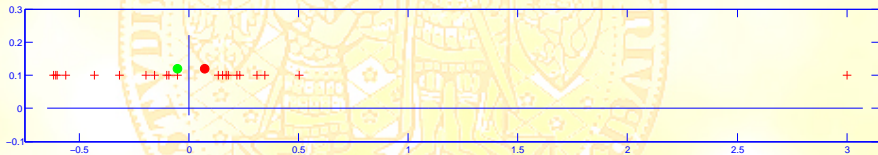
Contamination at point 2



First of all some inspiration

Observe the mean ● and the median ●

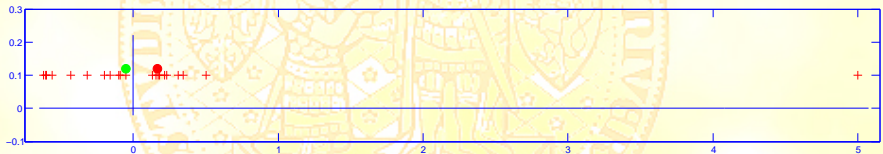
Contamination at point 3



First of all some inspiration

Observe the mean ● and the median ●

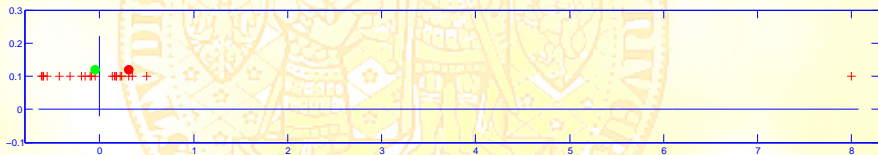
Contamination at point 5



First of all some inspiration

Observe the mean ● and the median ●

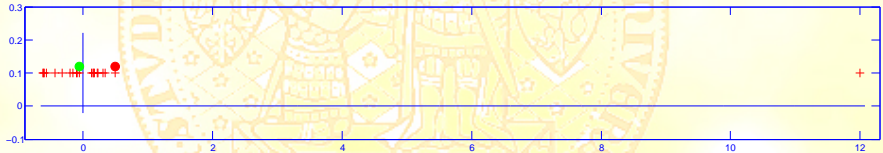
Contamination at point 8



First of all some inspiration

Observe the mean ● and the median ●

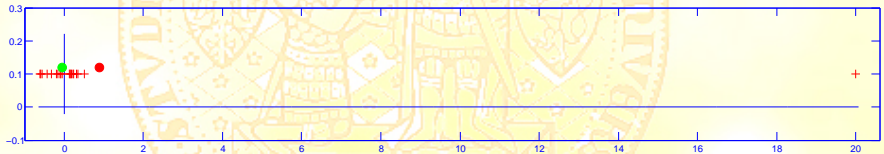
Contamination at point 12



First of all some inspiration

Observe the mean ● and the median ●

Contamination at point 20



First of all some inspiration

Observe the mean ● and the median ●

Contamination at point 20

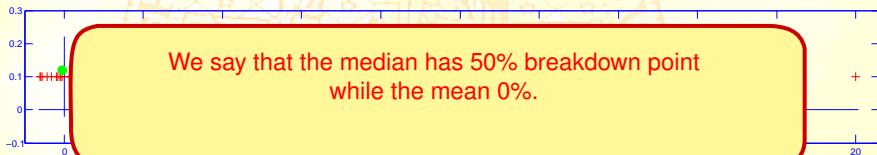


We would have to move 50% of observations
to see any impact on the median.

First of all some inspiration

Observe the mean ● and the median ●

Contamination at point 20

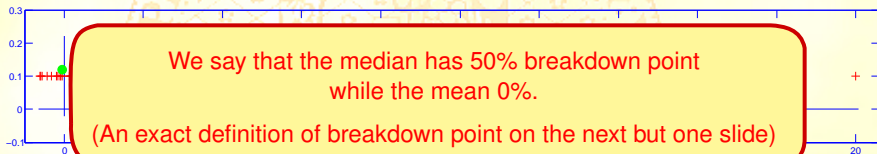


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First of all some inspiration

Observe the mean ● and the median ●

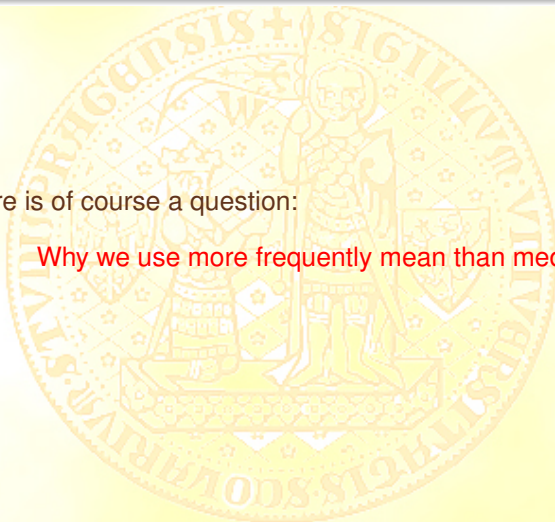
Contamination at point 20



We would have to move 50% of observations to see any impact on the median.

Then there is of course a question:

Why we use more frequently mean than median?



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Why we use more frequently mean than median?

$$\lim_{n \rightarrow \infty} \frac{\text{var}_{\Phi}(\bar{x}^{(n)})}{\text{var}_{\Phi}(\text{median}^{(n)})} = 0.6\dots$$

Hampel's approach - characteristics of the functional T at the d. f. F

- An overall characteristic of the functional (the estimator) is

$$\varepsilon^* = \sup \{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, K_\varepsilon \text{ compact} \}$$

$$\left. \pi(F, G) < \varepsilon \Rightarrow G(\{T_n \in K_\varepsilon\}) \xrightarrow{n \rightarrow \infty} 1 \right\}$$

where $\pi(F, G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$ and T_n is an empirical counterpart to the functional T .

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- ε^* is called *breakdown point*

(explanation of Prokhorov metric is on one of the next slides, then finite sample breakdown point).

Hampel's approach - characteristics of the functional T at the d. f. F

- An overall characteristic of the functional (the estimator) is

 ε^*

It is not trivial to understand this definition -
- we shall try after some preparation.

where $\rho(F, G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$
and T_n is an empirical counterpart to the functional T .

- ε^* is called *breakdown point*.

Hampel's approach - characteristics of the functional T at the d. f. F

An overall characteristic of the functional (the estimator) is

The definition contains new notions:

compact set and Prokhorov metric.

ϵ^* is called breakdown point.

Hampel's approach - characteristics of the functional T at the d. f. F

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- 1 Firstly, what is a compact set? And what is good for ?

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Hampel's approach - characteristics of the functional T at the d. f. F

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- 1 Firstly, what is a compact set? And what is good for ?
- 2 Secondly, an inspiration why we need Prokhorov metric.

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An **overall characteristic of the functional** (the estimator) is

The definition contains new notions:

compact set and **Prokhorov metric**.

- 1 Firstly, what is a compact set? And what is good for ?
- 2 Secondly, an inspiration why we need Prokhorov metric.
- 3 Thirdly, an explanation what Prokhorov metric is.

ϵ^* is called **breakdown point**.

Hampel's approach - characteristics of the functional T at the d. f. F

- An overall characteristic of the functional (the estimator) is

Then we return to the definition of **breakdown point**.

where $\rho(T, G)$ is the *Prokhorov metric* of $T(x)$ and $G(x)$
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Hampel's approach - characteristics of the functional T at the d. f. F

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Hampel's approach - characteristics of the functional T at the d. f. F

- An **overall characteristic of the functional** (the estimator) is

Then we return to the definition of **breakdown point**.

- 1 Firstly, we explain the sense of it.
- 2 Then we try to read the mathematics.

where $\rho(T, G)$ is the *Frobenius metric* of $T(x)$ and $G(x)$
and T_n is an empirical counterpart to the functional T .

- ε^* is called **breakdown point**.

So, let us start with compact set

- Open and closed sets

C is closed set if for any sequence $\{x_n\}_{n=1}^{\infty} \subset C$ such that

$$\exists \left(\lim_{n \rightarrow \infty} x_n = x_0 \right) \Rightarrow x_0 \in C.$$

So, let us start with compact set

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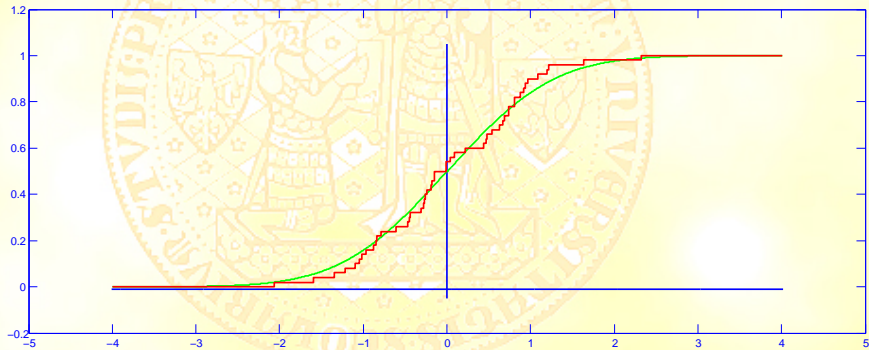
$$\exists \left(\lim_{n \rightarrow \infty} x_n = x_0 \right) \Rightarrow x_0 \in C.$$

- C is compact if it is closed and $\forall (x \in C) \|x\| < K < \infty$.
- The sense of compact sets and the use of compactness:

Any open cover contains a finite subcover.

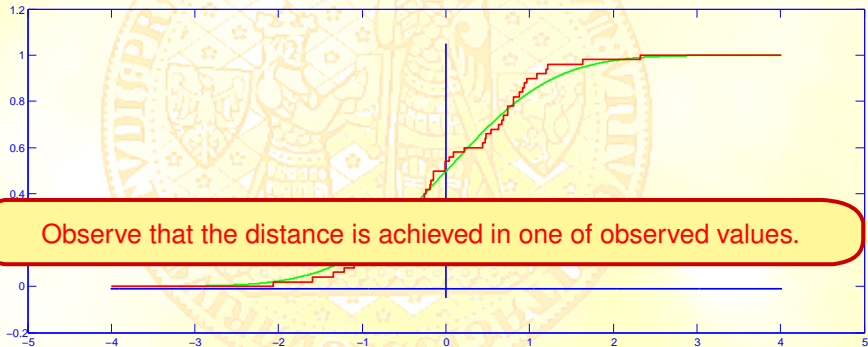
Recalling Kolmogorov-Smirnov metric

$$D(F, G) = \sup_{-\infty < x < \infty} |F(x) - G(x)|$$



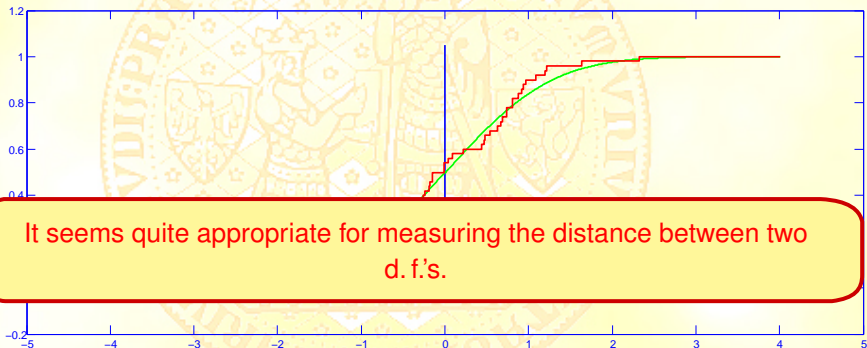
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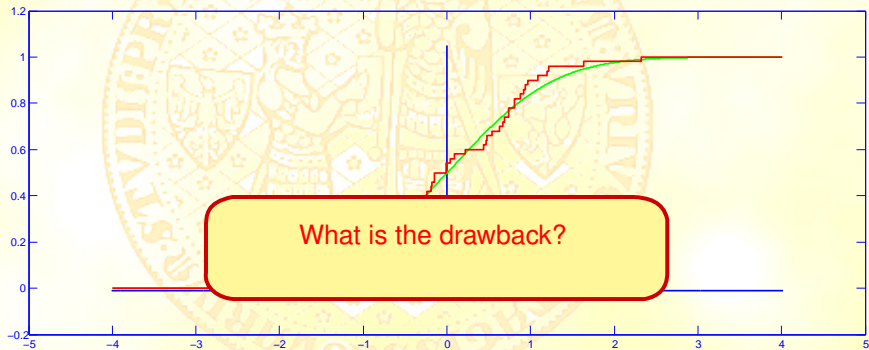
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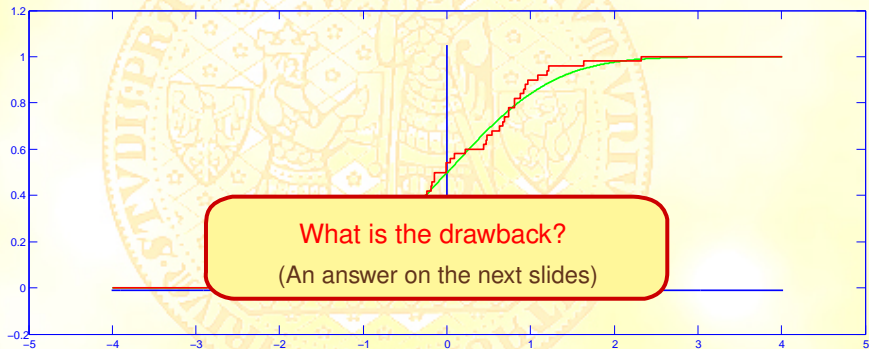
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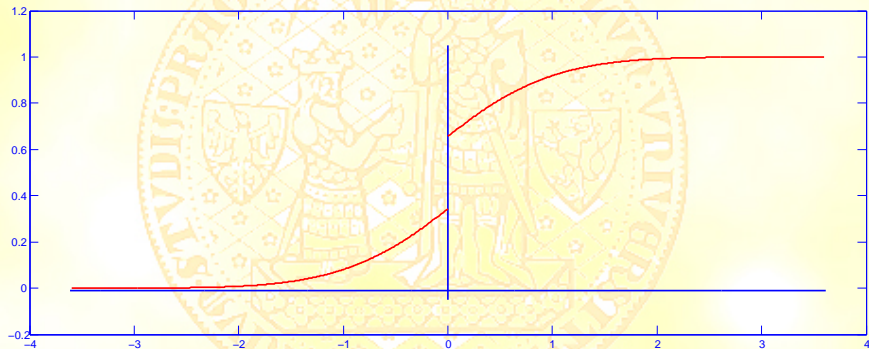
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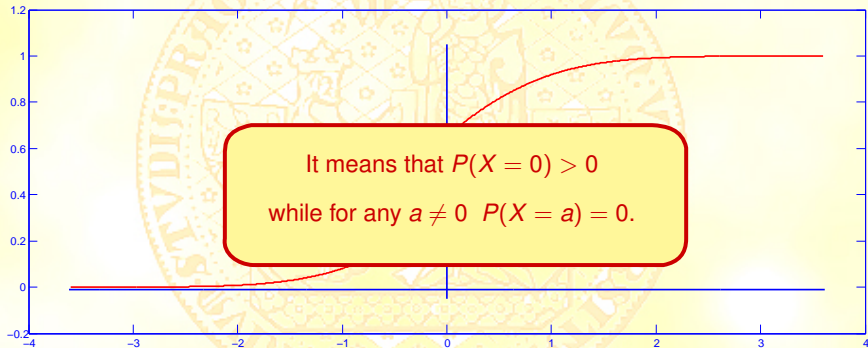
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Distribution function with a jump in the origine



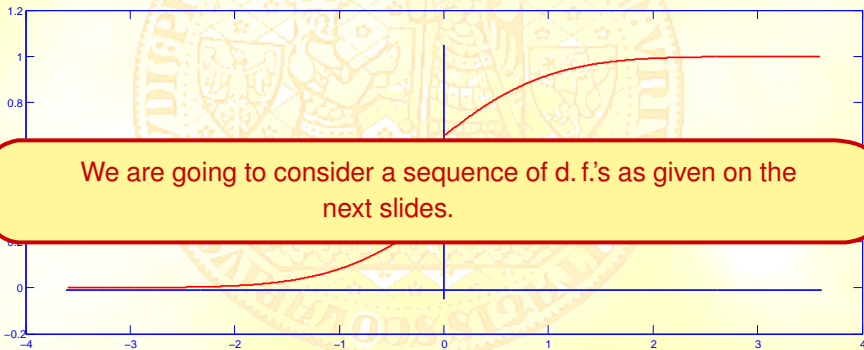
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

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Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

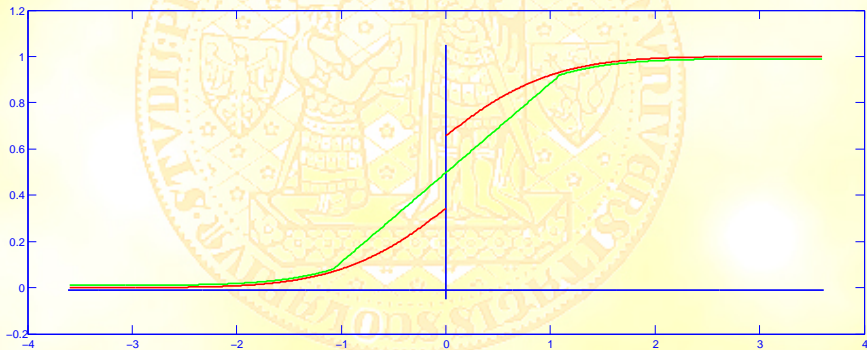
D. f. with a jump in the origine .



We are going to consider a sequence of d. f.'s as given on the next slides.

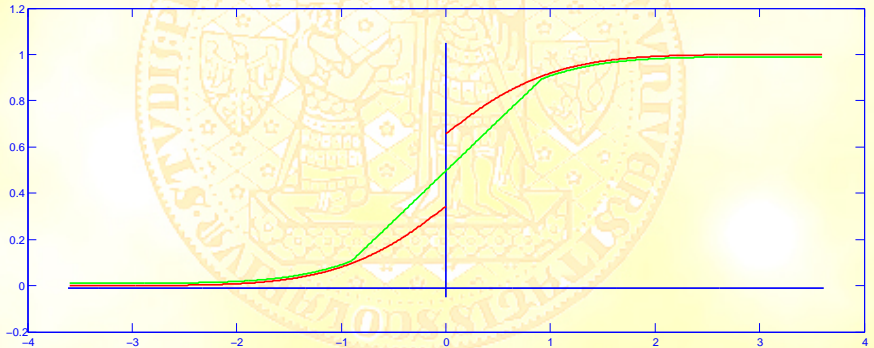
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 2.2 .



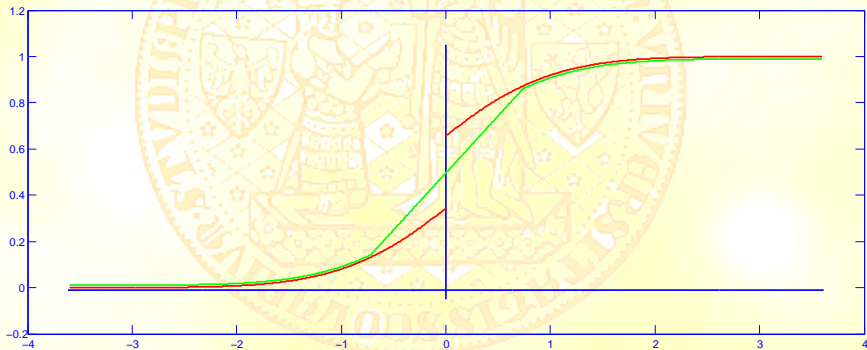
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 1.8 .



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

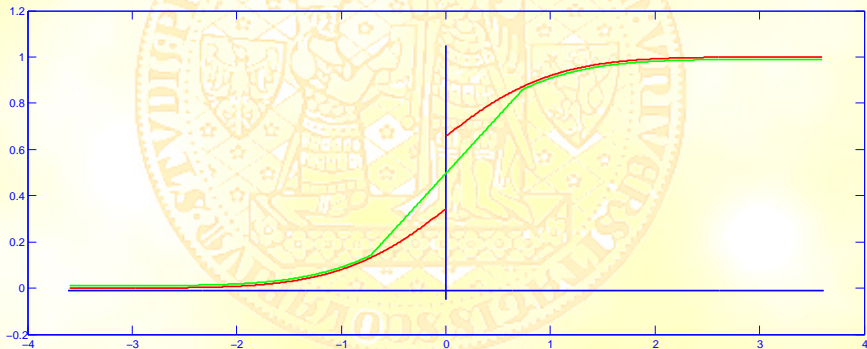
Width of linear part 1.5.



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

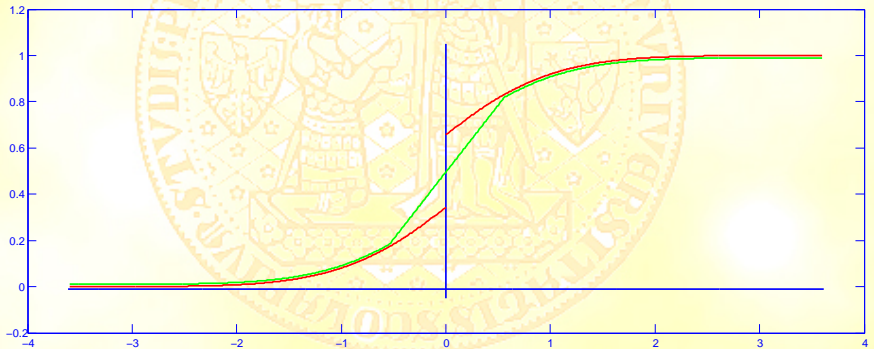
Width of linear part 1.5.

Could You say what is the K-S distance between green and red d. f.?



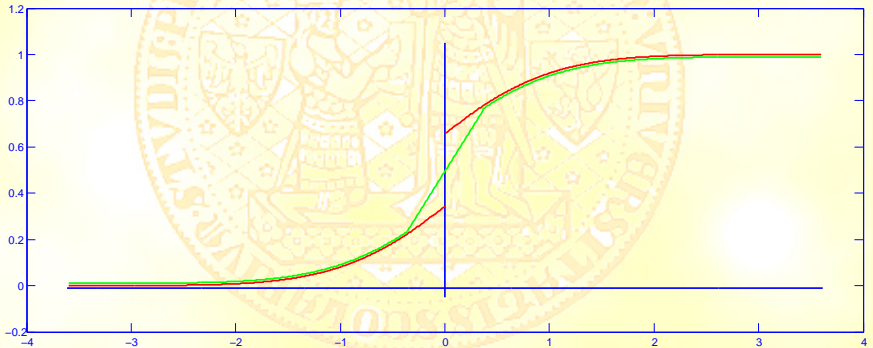
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 1.1.



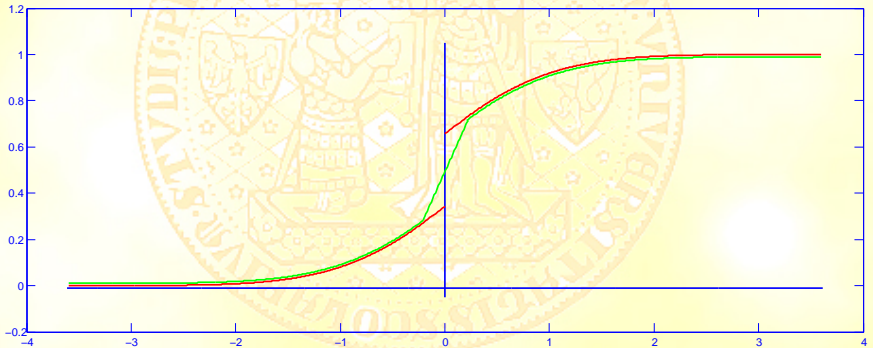
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.8.



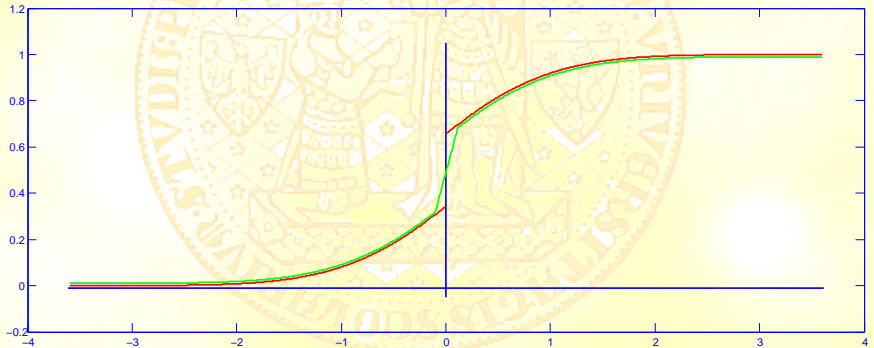
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.4.



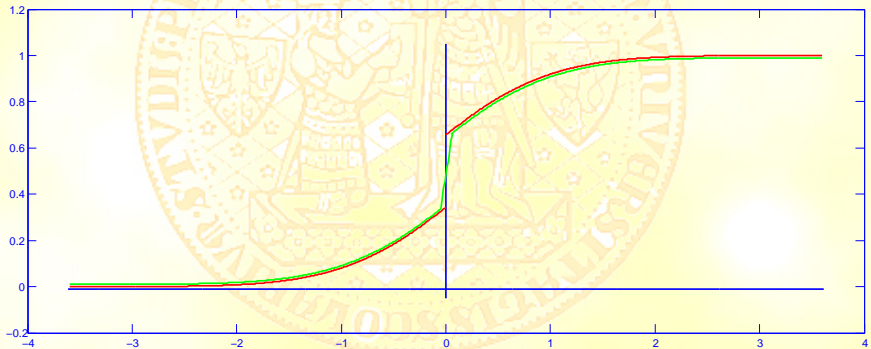
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.3.



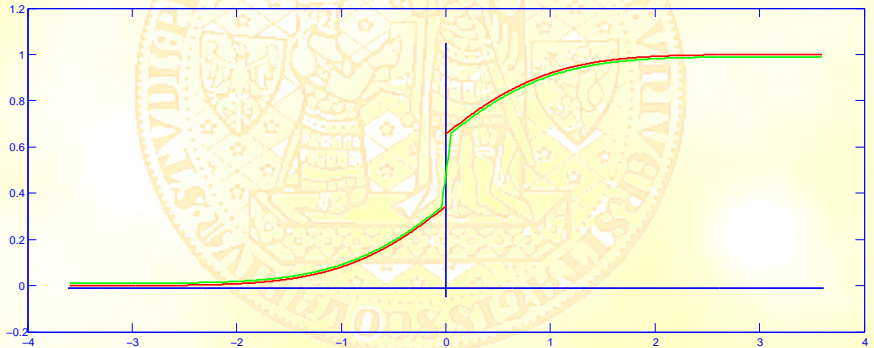
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.2.



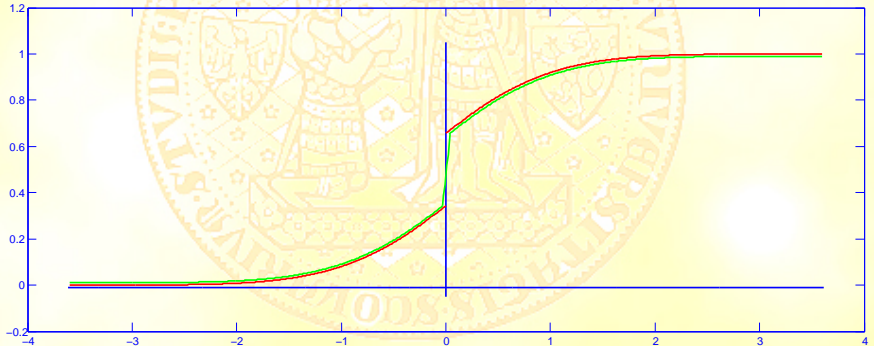
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.1.



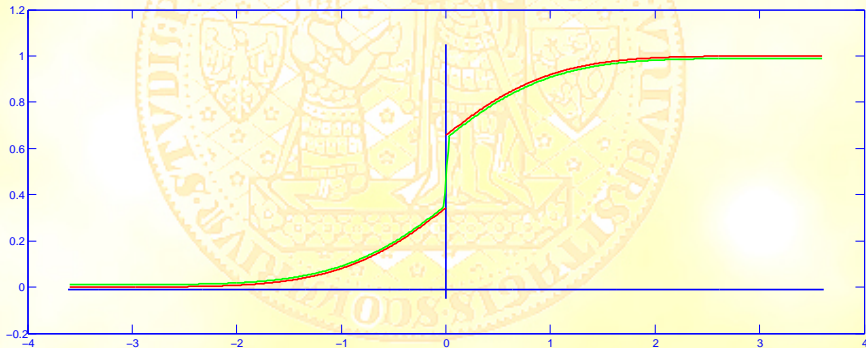
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.08.



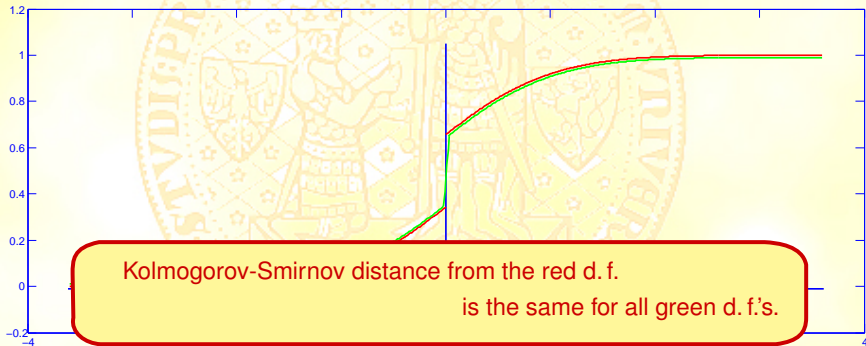
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.06.



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

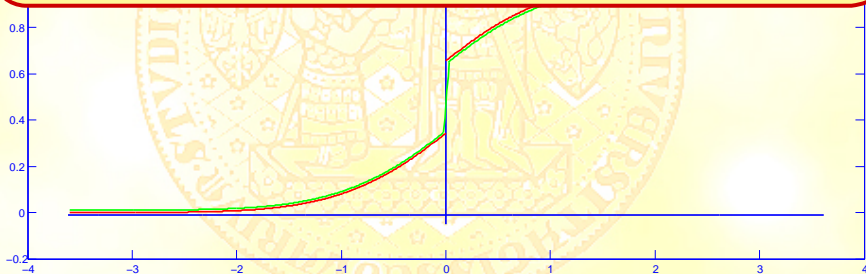
Width of linear part 0.06.



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

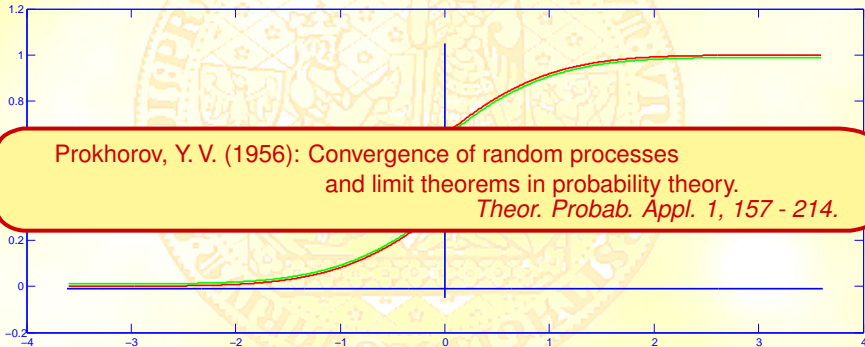
Width of linear part 0.06.

Kolmogorov-Smirnov distance need not be appropriate
for measuring distance between d.f.'s in all situations.



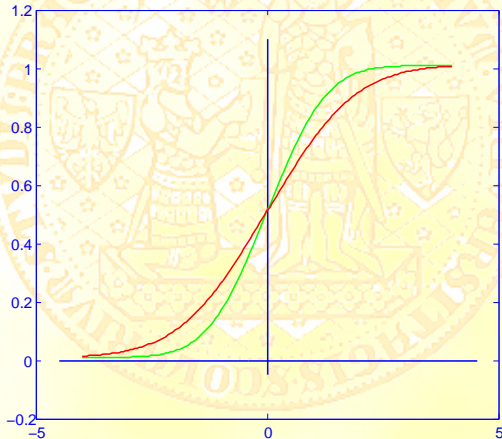
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.06.



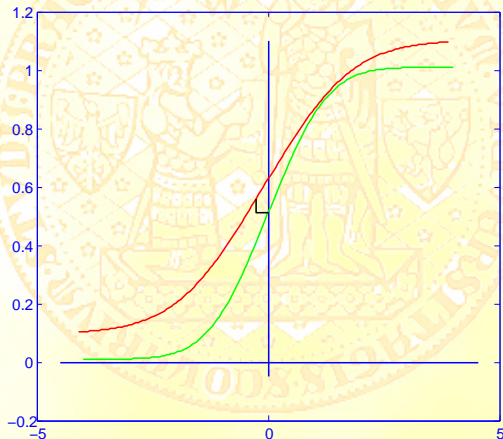
Explaining Prohorov distance (notice different transcription)

Coinsider two d. f.'s



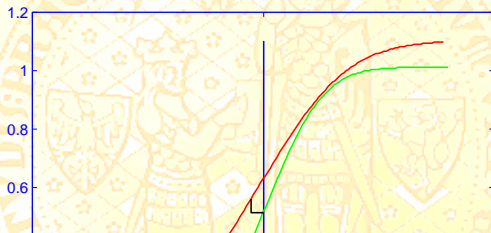
Explaining Prokhorov distance

Prokhorov metric

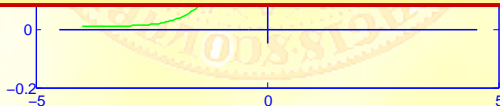


Explaining Prokhorov distance

Prokhorov metric

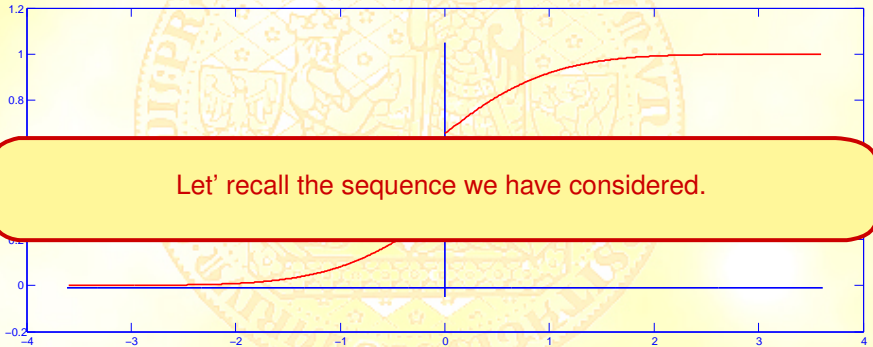


$$\pi(F, G) = \inf \{ \varepsilon \in [0, 1] : \forall (x \in \mathbb{R}) \quad F(x) \leq G(x + \varepsilon) + \varepsilon \vee G(x) \leq F(x + \varepsilon) + \varepsilon \}$$



Comparing Kolmogorov-Smirnov and Prokhorov metrics

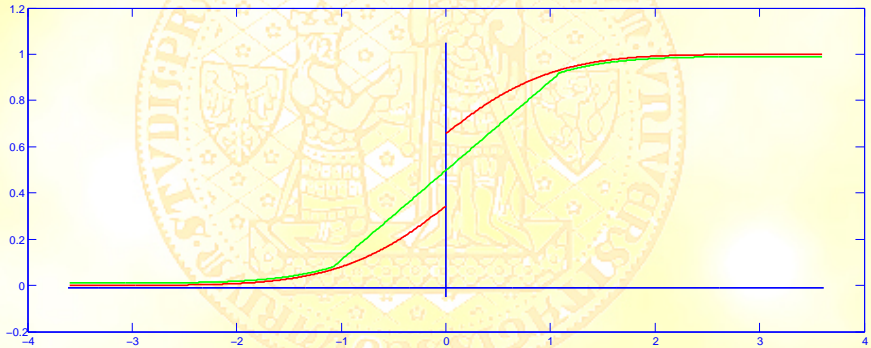
D. f. with a jump in the origine .



Let' recall the sequence we have considered.

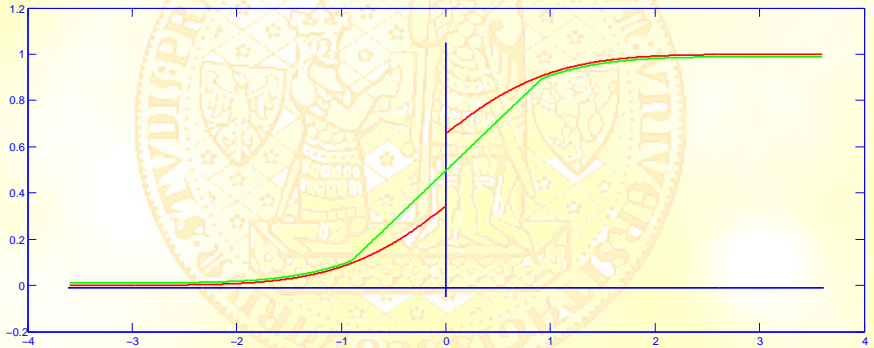
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 2.2 .



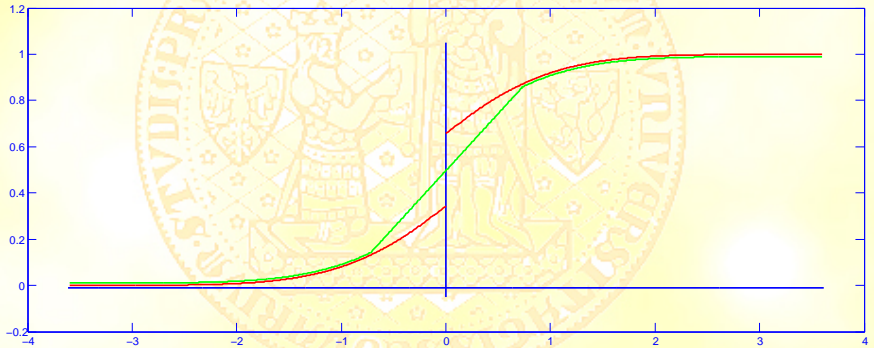
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 1.8 .



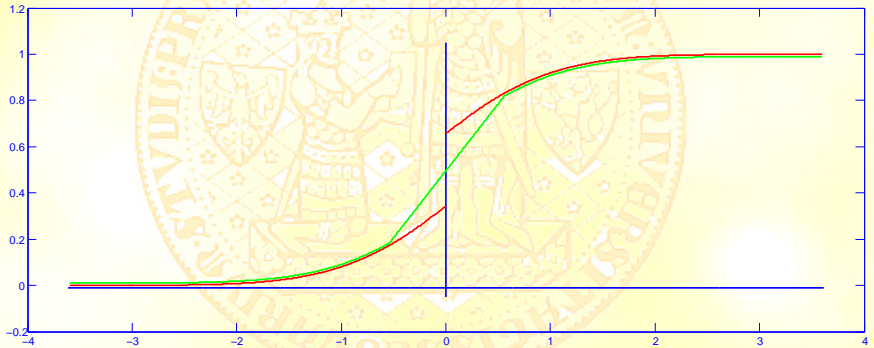
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 1.5.



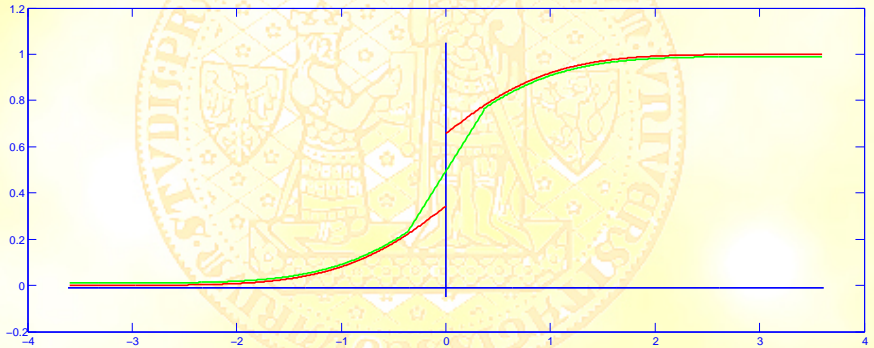
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 1.1.



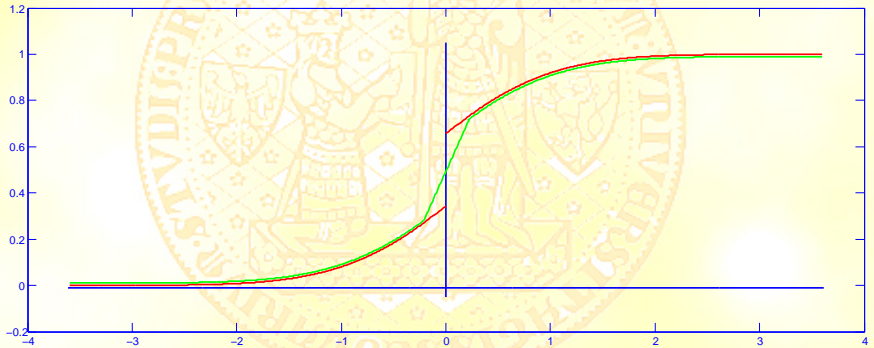
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.8.



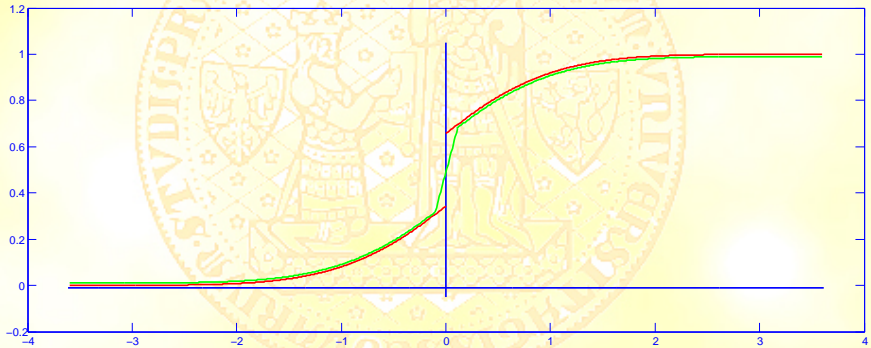
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.4.



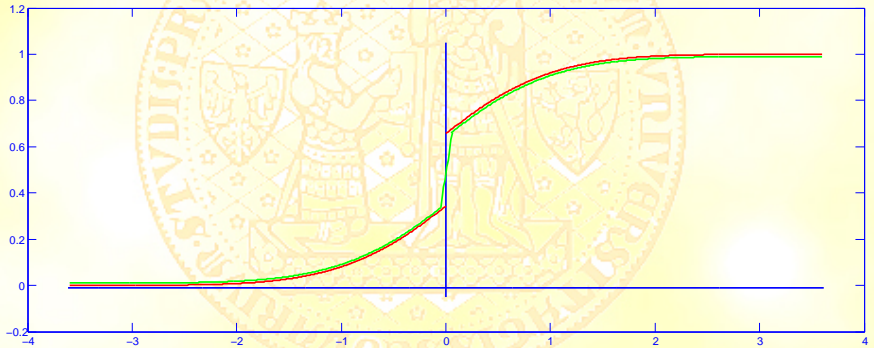
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.3.



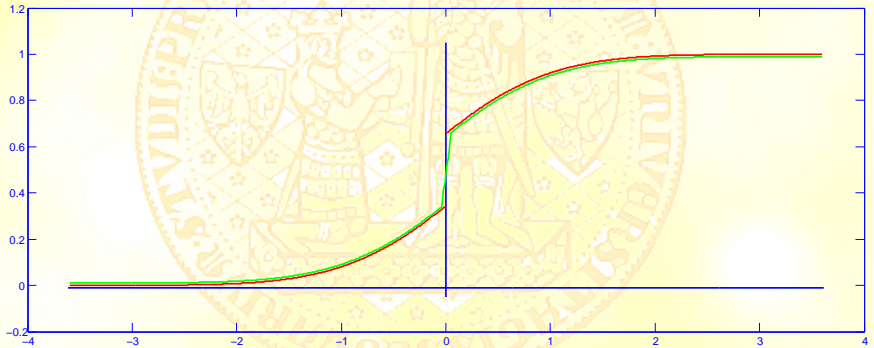
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.2.



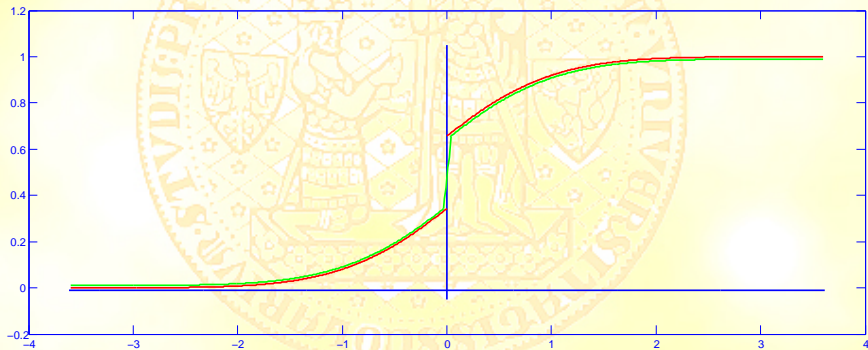
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.1.



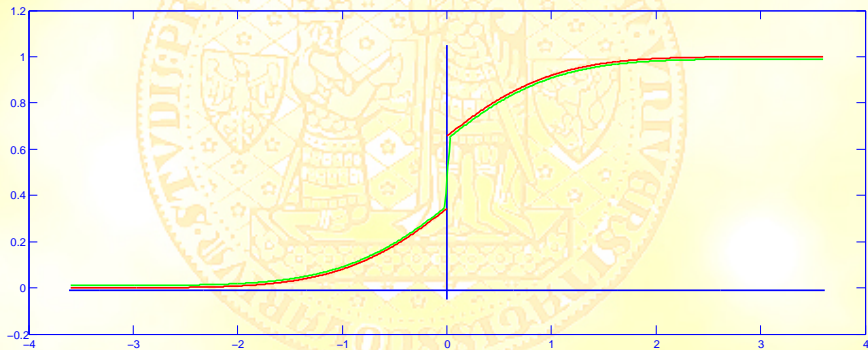
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.08.



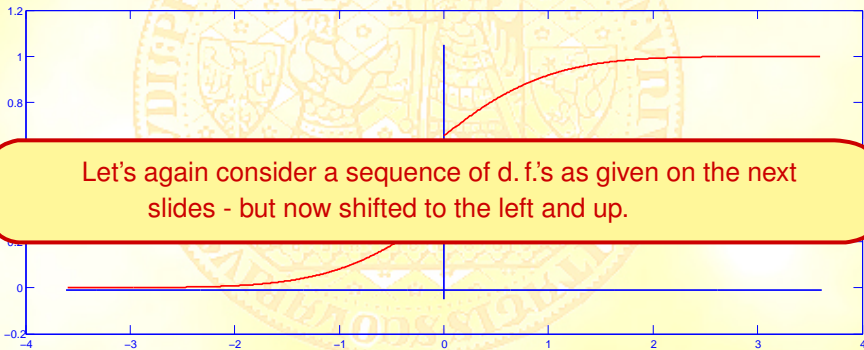
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.06.



The convergence in Prokhorov metrics corresponds with intuitive idea of it.

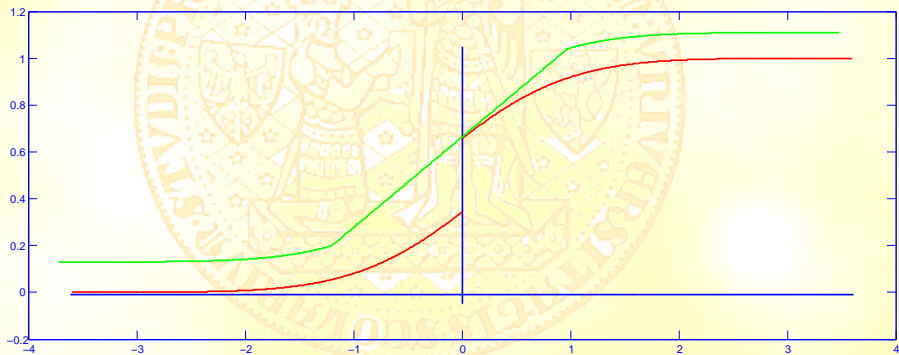
D. f. with a jump in the origine .



Let's again consider a sequence of d. f.'s as given on the next slides - but now shifted to the left and up.

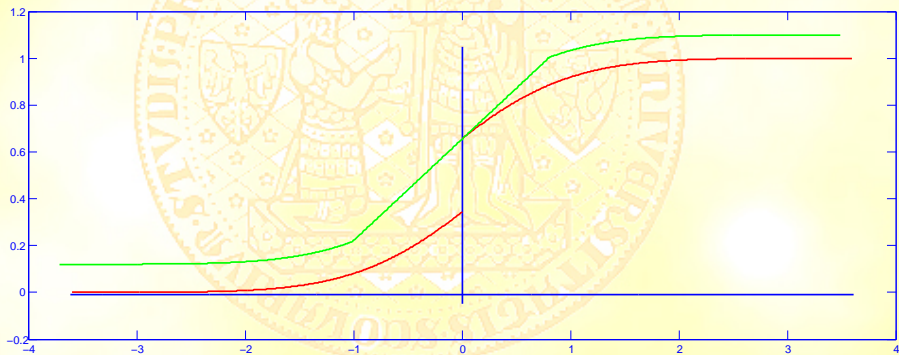
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 2.2 .



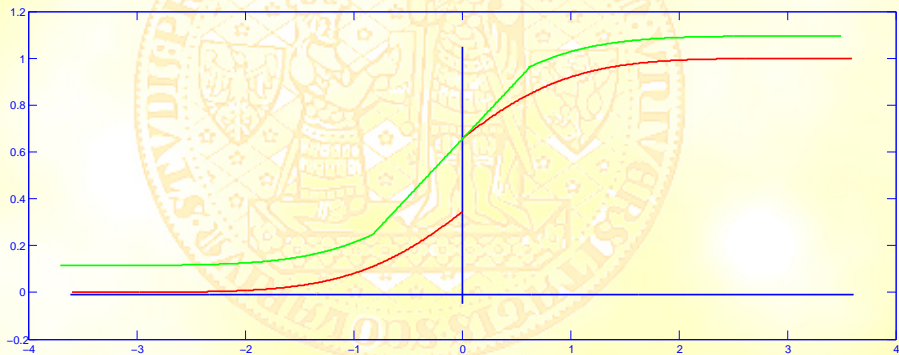
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 1.8 .



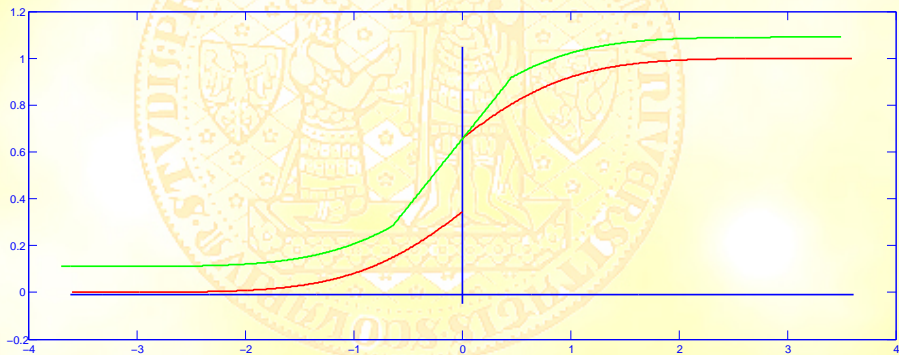
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 1.5.



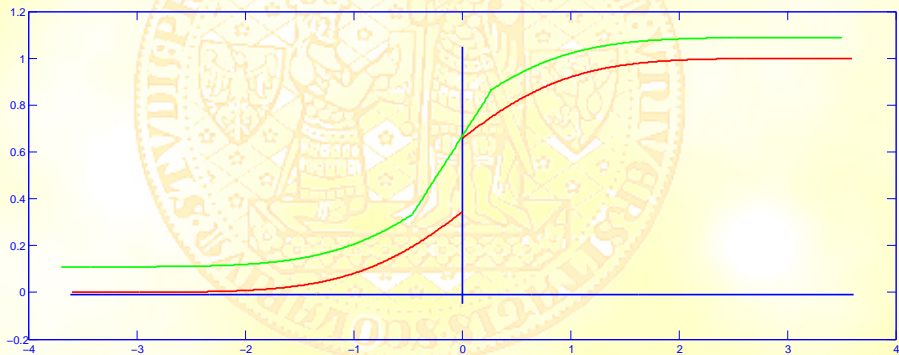
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 1.1.



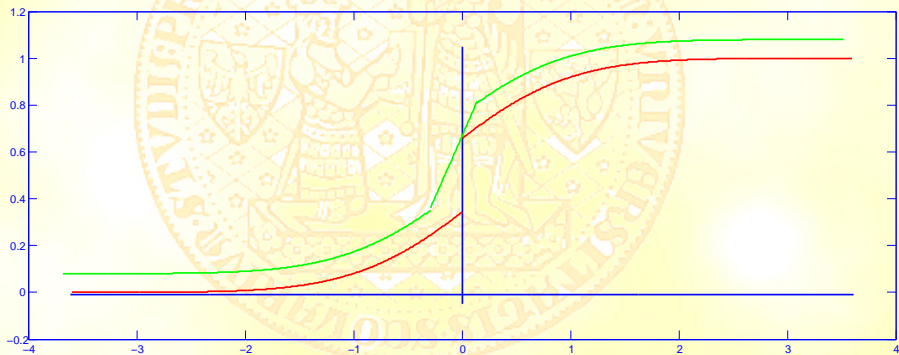
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.8.



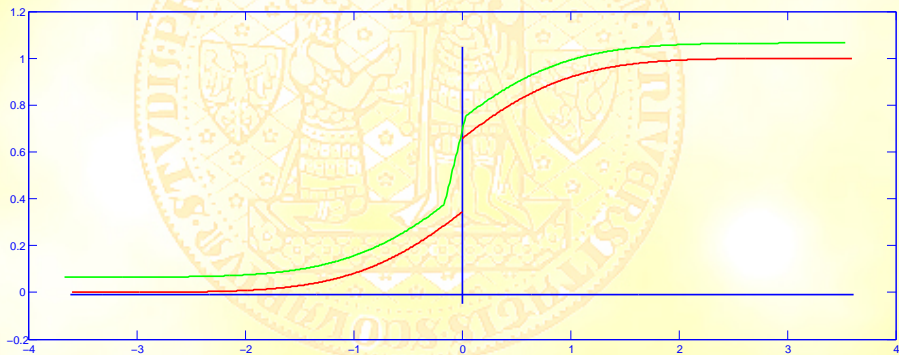
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.4.



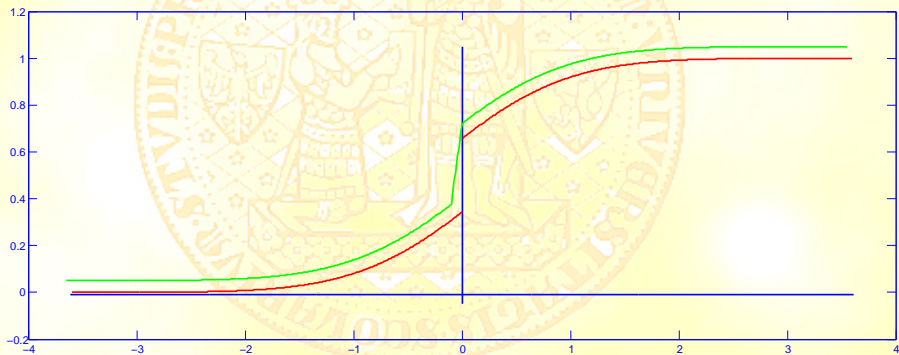
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.3.



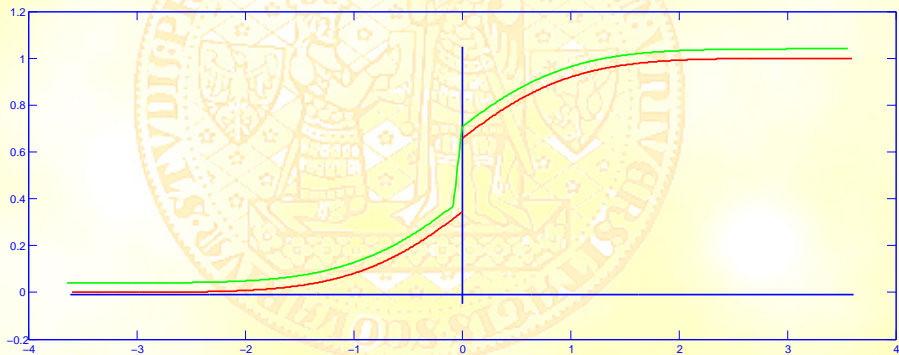
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.2.



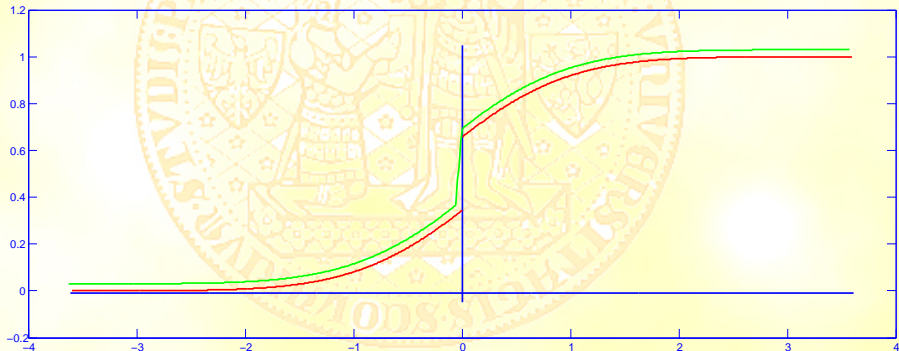
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.1.



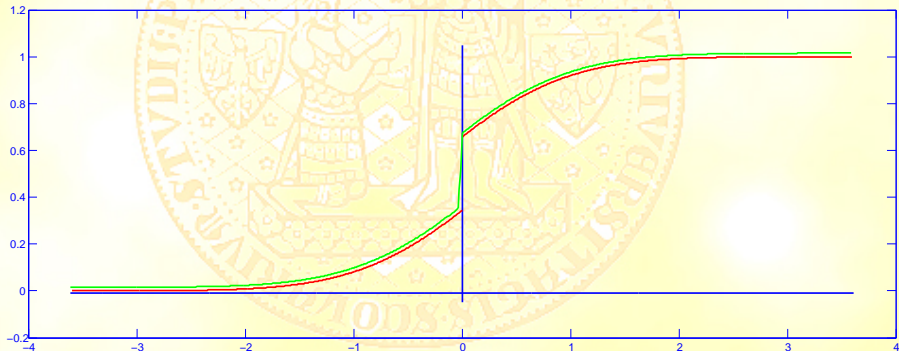
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.08.



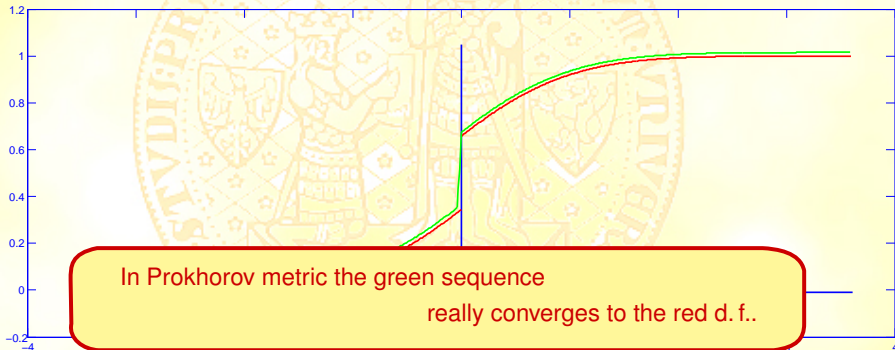
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.06.



The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.06.



The global empirical characteristic of estimator.

Hampel's approach - characteristics of the functional T at the d. f. F

- Breakdown point - "finite" sample version

$$x_1, x_2, \dots, x_n \Rightarrow T_n(x_1, x_2, \dots, x_n)$$

The global empirical characteristic of estimator.

Hampel's approach - characteristics of the functional T at the d. f. F

- Breakdown point - "finite" sample version

$$x_1, x_2, \dots, x_n \Rightarrow T_n(x_1, x_2, \dots, x_n)$$

- Find maximal m_n such that for any

$$y_1, y_2, \dots, y_{m_n} \Rightarrow |T_n(x_1, x_2, \dots, x_{n-m_n}, y_1, y_2, \dots, y_{m_n})| < \infty$$

$$(0 < T_n(x_1, x_2, \dots, x_{n-m_n}, y_1, y_2, \dots, y_{m_n}) < \infty \text{ - for scale }).$$

The global empirical characteristic of estimator.

Hampel's approach - characteristics of the functional T at the d. f. F

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- Put

$$\varepsilon^* = \lim_{n \rightarrow \infty} \frac{m_n}{n}$$

(we'll return to it later on,

now let's return to the exact definition of the **breakdown point**).

Let's read the definition of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

- An **overall characteristic of the functional** (the estimator) is

$$\varepsilon^* = \sup \{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, K_\varepsilon \text{ compact} \}$$

$$\pi(F, G) < \varepsilon \Rightarrow G(\{T_n \in K_\varepsilon\}) \xrightarrow{n \rightarrow \infty} 1 \}$$

where $\pi(F, G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$
and T_n is an empirical counterpart to the functional T .

Let's read the definition of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

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- ε^* is called **breakdown point**.

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Let's rewrite the mathematical part of definition on the next slide.

Let's read the definition of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup \{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, K_\varepsilon \text{ compact}$$

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Let's read the definition of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup \{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, K_\varepsilon \text{ compact} \}$$

$$\left. \pi(F, G) < \varepsilon \Rightarrow G(\{T_n \in K_\varepsilon\}) \xrightarrow{n \rightarrow \infty} 1 \right\}$$

- 1 Let's assume that $T \in R$, so it is connected with a parameter of F .

Let's read the definition of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup \{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, K_\varepsilon \text{ compact}$$

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- ① Let's assume that $T \in R$, so it is connected with a parameter of F .
- ② K_ε could be sufficiently wide interval - for this moment.

Let's read the definition of breakdown point

Hampel's approach - characteristics of the functional T at the d.f. F

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- 1 Let's assume that $T \in R$, so it is connected with a parameter of F .
- 2 K_ε could be sufficiently wide interval - for this moment.
- 3 The definition says that T_n is so "good" for estimating " F " that whenever G is sufficiently close to F , T_n converges in probability with respect to G to something finite.

Let's read the definition of breakdown point

Hampel's approach - characteristics of the functional T at the d.f. F

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That's all.

Content

- 1 Repetition is mother of wisdom (Jan mos Komensk)
- 2 The breakdown point
- 3 Specification of robustness characteristics for classical estimators

Specifying the “robustness” characteristics for the location parameter

At the beginning of the third lecture

we have computed influence function for $T(\Phi) = E_{\Phi}Z$:

Recalling definition of influence function is :

Fix a functional $T : \mathcal{H} \rightarrow \mathcal{R} \dots$

$$IF(x, T, F) = \lim_{\delta \rightarrow 0} \frac{T\left((1 - \delta)F(\cdot) + \delta \cdot \Delta_x\right) - T\left(F(\cdot)\right)}{\delta}.$$

- 1 Fix $T(\Phi) = E_{\Phi}(Z) = \int Z d\Phi = \int z \cdot \phi(z) dz$.
- 2 $T(\Phi(\cdot)) = \frac{1}{\sqrt{2\pi}} \int z \cdot \exp\left\{-\frac{z^2}{2}\right\} dz = 0$
- 3 $T\left((1 - \delta)\Phi(\cdot) + \delta \cdot \Delta_x\right)$
 $= \int z \left\{ (1 - \delta) \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{z^2}{2}\right\} + \delta \cdot \Delta_x \right\} dz = (1 - \delta) \cdot 0 + \delta \cdot x.$
- 4 Finally, $IF(x, T, \Phi) = \lim_{\delta \rightarrow 0} \frac{\delta \cdot x}{\delta} = x.$

Specifying the “robustness” characteristics for the location parameter

We easily verify that the same computation can be done
whenever r. v. Z has finite mean value $T(F) = E_F Z = \mu \in R$:

Fix a functional $T : \mathcal{H} \rightarrow R \dots$

$$IF(x, T, F) = \lim_{\delta \rightarrow 0} \frac{T\left((1 - \delta)F(\cdot) + \delta \cdot \Delta_x\right) - T(F(\cdot))}{\delta}.$$

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- 1 Fix $T(F) = E_F(Z) = \int Z d\Phi = \int z \cdot f(z) dz$.

Specifying the “robustness” characteristics for the location parameter

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- 2 $T(F(\cdot)) = \int z \cdot f(z) dz = \mu$

Specifying the “robustness” characteristics for the location parameter

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- 3 $T\left((1 - \delta)F(\cdot) + \delta \cdot \Delta_x\right)$
 $= \int z \{(1 - \delta)f(x) + \delta \cdot \Delta_x\} dz = (1 - \delta) \cdot \mu + \delta \cdot x.$

Specifying the “robustness” characteristics for the location parameter

We easily verify that the same computation can be done
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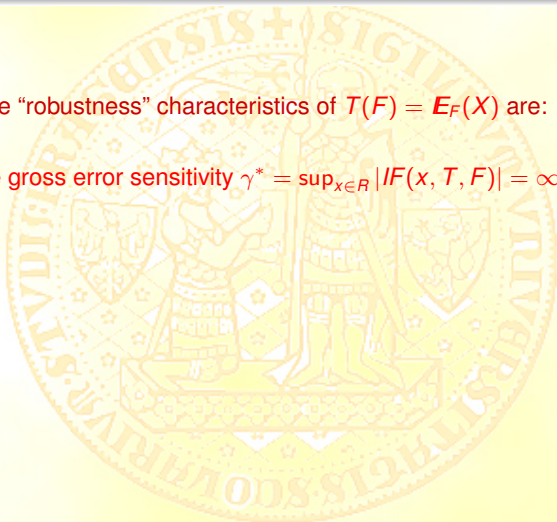
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 $= \int z \{(1 - \delta)f(x) + \delta \cdot \Delta_x\} dz = (1 - \delta) \cdot \mu + \delta \cdot x.$
- 4 Finally, $IF(x, T, F) = \lim_{\delta \rightarrow 0} \frac{\delta \cdot (-\mu + x)}{\delta} = -\mu + x.$

Specifying the “robustness” characteristics for the location parameter

Hence the “robustness” characteristics of $T(F) = E_F(X)$ are:

- 1 The gross error sensitivity $\gamma^* = \sup_{x \in R} |IF(x, T, F)| = \infty$.



Specifying the “robustness” characteristics for the location parameter

Hence the “robustness” characteristics of $T(F) = E_F(X)$ are:

- 1 The gross error sensitivity $\gamma^* = \sup_{x \in R} |IF(x, T, F)| = \infty$.
- 2 The local-shift sensitivity $\lambda^* = \sup_{x, y \in R} \left| \frac{IF(x, T, F) - IF(y, T, F)}{x - y} \right| = 1$.

Specifying the “robustness” characteristics for the location parameter

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Specifying the “robustness” characteristics for the location parameter

Hence the “robustness” characteristics of $T(F) = E_F(X)$ are:

- ① The gross error sensitivity $\gamma^* = \sup_{x \in R} |IF(x, T, F)| = \infty$.
- ② The local-shift sensitivity $\lambda^* = \sup_{x, y \in R} \left| \frac{IF(x, T, F) - IF(y, T, F)}{x - y} \right| = 1$.
- ③ The rejection point $\rho^* = \inf \{ r \in R^+ : IF(x, T, F) = 0, |x| > r \} = \infty$.
- ④ **The breakdown point $\varepsilon^* = 0$**

(the last characteristic is “derived heuristically”

from the finite version of breakdown point).

Specifying the “robustness” characteristics for the scale parameter

At the beginning of the third lecture

we have also computed influence function for $T(\Phi) = E_{\Phi} X^2$:

The **Recalling again definition of influence function is** :

Fix a functional $T : \mathcal{H} \rightarrow R \dots$

$$IF(x, T, F) = \lim_{\delta \rightarrow 0} \frac{T\left((1 - \delta)F(\cdot) + \delta \cdot \Delta_x\right) - T\left(F(\cdot)\right)}{\delta}.$$

① Fix $T(\Phi) = E_{\Phi}(X^2) = \int X^2 d\Phi = \int z^2 \cdot \phi(z) dz$.

② $T(\Phi(\cdot)) = \frac{1}{\sqrt{2\pi}} \int z^2 \cdot \exp\left\{-\frac{z^2}{2}\right\} dz = 1$

③ $T\left((1 - \delta)\Phi(\cdot) + \delta \cdot \Delta_x\right)$
 $= \int z^2 \left\{ (1 - \delta) \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{z^2}{2}\right\} + \delta \cdot \Delta_x \right\} dz = (1 - \delta) \cdot 1 + \delta \cdot x^2.$

④ Finally, $IF(x, T, \Phi) = \lim_{\delta \rightarrow 0} \frac{(1 - \delta) \cdot 1 + \delta \cdot x^2 - 1}{\delta} = -1 + x^2.$

Specifying the “robustness” characteristics for the scale parameter

We easily verify that the same computation can be done
whenever r. v. Z has finite variance $T(F) = E_F (Z - EZ)^2 = \sigma^2 \in R^+$:

Fix a functional $T : \mathcal{H} \rightarrow R \dots$

$$IF(x, T, F) = \lim_{\delta \rightarrow 0} \frac{T\left((1 - \delta)F(\cdot) + \delta \cdot \Delta_x\right) - T(F(\cdot))}{\delta}.$$

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- ① Fix $T(F) = E_F(Z - EZ)^2 = \int (Z - EZ)^2 dF = \int (z - EZ)^2 \cdot f(z) dz.$

Specifying the “robustness” characteristics for the scale parameter

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- 2 $T(F(\cdot)) = \int (z - EZ)^2 \cdot f(z) dz = \sigma^2$

Specifying the “robustness” characteristics for the scale parameter

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whenever r. v. Z has finite variance $T(F) = \mathbf{E}_F(Z - \mathbf{E}Z)^2 = \sigma^2 \in \mathbb{R}^+$:

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- 2 $T(F(\cdot)) = \int (z - \mathbf{E}Z)^2 \cdot f(z) dz = \sigma^2$
- 3 $T\left((1 - \delta)F(\cdot) + \delta \cdot \Delta_x\right)$
 $= \int (z - \mathbf{E}Z)^2 \{(1 - \delta)f(z) + \delta \cdot \Delta_x\} dz = (1 - \delta) \cdot \sigma^2 + \delta \cdot (x - \mathbf{E}Z)^2.$

Specifying the “robustness” characteristics for the scale parameter

We easily verify that the same computation can be done

whenever r. v. Z has finite variance $T(F) = \mathbf{E}_F(Z - \mathbf{E}Z)^2 = \sigma^2 \in \mathbb{R}^+$:

Fix a functional $T : \mathcal{H} \rightarrow \mathbb{R} \dots$

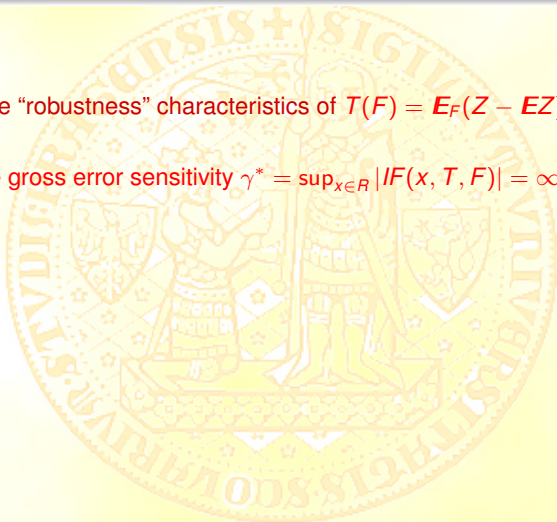
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- 3 $T\left((1 - \delta)F(\cdot) + \delta \cdot \Delta_x\right)$
 $= \int (z - \mathbf{E}Z)^2 \{(1 - \delta)f(z) + \delta \cdot \Delta_x\} dz = (1 - \delta) \cdot \sigma^2 + \delta \cdot (x - \mathbf{E}Z)^2$.
- 4 Finally, $IF(x, T, F) = \lim_{\delta \rightarrow 0} \frac{\delta \cdot (-\sigma^2 + (x - \mathbf{E}Z)^2)}{\delta} = -\sigma^2 + (x - \mathbf{E}Z)^2$.

Specifying the “robustness” characteristics for the scale parameter

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- ④ **The breakdown point $\varepsilon^* = 0$**

(the last characteristic is again “derived heuristically”
from the finite version of breakdown point).



THANKS FOR ATTENTION