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The Efficiency of Fairness in Voting Systems

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Abstract Fair representation of voters in a committee representing different voters' groups is being broadly discussed during last few years. Assuming we know what the fair representation is, there exists a problem of optimal quota: given a 'fair' distribution of voting weights, how to set up voting rule (quota) in such a way that distribution of relative a priori voting power is as close as possible to distribution of relative voting weights. Together with optimal quota problem a problem of trade-off between fairness and efficiency (ability of a voting body to change status quo) is formalized by a fairness-efficiency matrix.

Keywords Committee system, efficiency, fairness, fairness-efficiency matrix, indirect voting power, optimal quota, power indices, voting system

1. Introduction

Let us consider n units (e.g. regions, political parties) with different size of population (voters), represented in a super-unit committee that decides different agendas relevant for the whole entity. Each unit representation in the committee has some voting weight (number of votes). By *voting system* we mean an allocation of voting weights in elections and committees, the form of the ballot and rules for counting the votes to determine outcome of voting.

Voting weight is not the same thing as voting power. Usually voting power means an ability to influence outcome of voting. Voting power indices are used to evaluate a probability that a particular voter is 'decisive in voting' in the sense that if her vote is 'yes', then the outcome of voting in committee is 'yes', and if she votes 'no', the outcome is 'no'.

Two aspects of voting systems are being discussed: fairness and efficiency. While the fairness is related to distribution of voting power of different actors

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of voting, efficiency is an ability of the system to change status quo.

Concept of *fairness* is usually based on the following rather artificial construction: Decision making process is performed by series of referenda in each unit and units' representations in the committee are voting according results of referenda. In each unit an individual citizen has one vote that provides him with a voting power (each citizen from one unit has the same voting power). Each unit representation has some voting power in the super-unit committee that follows from its voting weight in the committee. Indirect voting power of a citizen from particular unit is given by product of her voting power in local referenda and voting power of her unit representation in the committee. Fair representation of units in the super-unit committee means that each citizen has the same indirect voting power independently of the unit he belongs to. Concept of *efficiency* is based on a probability that a proposal will be passed in the committee.

Extension of the European Union and related changes in its voting system became a new impulse in discussions about fairness and efficiency. In the late spring of 2004 the open letter of European scientists to the governments of the EU member states was distributed in European academic community.[1] The basic idea of the proposal supported by the open letter is the following concept of 'fairness':

> If the European Union is a union of citizens, then it is fair when each citizen (independently on her national affiliation) exercises the same influence over the union issues. It is achieved when voting weight of each national representation in Council of Ministers is proportional to the square root of population.

So called square root rule is attributed to British statistician Lionel Penrose (1946) and is closely related to indirect voting power measured by Penrose-Banzhaf power index. Different aspects of square root rule had been analysed in Felsenthal and Machover (1998), 2004), Laruelle and Widgrén (1998), Baldwin and Widgrén (2004), Turnovec (2009). Square root rule 'fairness' in the EU Council of Ministers voting was discussed and evaluated in Felsenthal and Machover (2007), Słomczy ński and Życzkowski (2006, 2007), Leech and Aziz (2008) and others. Concept of efficiency is attributed to Coleman (1971) so called 'power of collectivity to act' (application to Council of Ministers voting see in Hosli (2008), Leech and Aziz (2008)).

This paper is not focused particularly on European Union. Assuming,

[1] The Open Letter was originally signed by the group of nine distinguished scientists from the six EU countries, calling themselves 'Scientists for a democratic Europe', later cosigned by 38 other colleagues, and submitted to the governments of member states and to Commission. The letter (including tables with results for EU of 25 members) and list of its signatories see e.g. at the following web address: http://www.esi2.us.es/~mbilbao/pdffiles/letter.pdf.

that a principle of fairness is selected for a distribution of voting weights, we are addressing the question how to achieve equality of voting power (at least approximately) to fair voting weights with a 'reasonable' level of efficiency. The concepts of strictly proportional power introduced by Berg and Holler (1986) and of optimal quota of Słomczy ński and Życzkowski (2007) are used to find, given voting weights, a quota minimizing a distance between actors' voting weights and their power indices.

In the second section basic definitions are introduced and used power indices methodology shortly resumed. Third section introduces concept of quota intervals of stable power and optimal quota, and a trade-off between fairness and efficiency, represented by a fairness-efficiency matrix is discussed. While the framework of analysis of fairness and efficiency is usually restricted to Penrose-Banzhaf concept of power, we are treating it in a more general framework and our results are relevant for any power index based on pivots or swings and for any concept of fairness.

2. Committees and Voting Power

A *Simple weighted committee* is a pair $[N, \mathbf{w}]$, where N be a finite set of n committee members$i = 1, 2, ..., n$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a nonnegative vector of committee members' voting weights (e.g. votes or shares). By 2^N we denote power set of N (set of all subsets of N). By voting configuration we mean an element $S \in 2^N$, subset of committee members voting uniformly ('yes' or 'no'), and $w(S) = \sum_{i \in S} w_i$ denotes voting weight of configuration S. The voting rule is defined by a quota q, satisfying $0 < q \leq w(N)$, where q represents the minimal total weight necessary to approve the proposal. The triple $[N, q, \mathbf{w}]$ is called a *simple quota weighted committee*. A voting configuration S in committee $[N, q, \mathbf{w}]$ is called winning if $w(S) \geq q$ and a losing in the opposite case.

Voting power analysis seeks an answer to the following question: given a simple quota weighted committee $[N, q, \mathbf{w}]$, what is an influence of its members over the outcome of voting? The absolute voting power of a member i is defined as a probability $\Pi_i[N, q, \mathbf{w}]$ such that i will be decisive in the sense that such situation appears in which she would be able to decide the outcome of voting by her vote (Nurmi (1997). The relative voting power is

$$\pi_i[N, q, \mathbf{w}] = \frac{\Pi_i[N, q, \mathbf{w}]}{\sum_{k \in N} \Pi_k[N, q, \mathbf{w}]}.$$

There are two basic concepts of decisiveness: a swing position as an ability of individual voter to change by unilateral switch from a 'yes' to 'no' outcome of voting, and pivotal position, such position of individual voter in a permutation of voters expressing ranking of attitudes of members to voted issue (from most preferable to least preferable) and corresponding order of forming of winning configuration, in which her vote 'yes' means a 'yes' outcome of the vote and her 'no' vote means a the outcome is 'no'.

Assuming many voting acts and all configurations are equally likely, it makes sense to evaluate the a priori voting power of each member of the committee by the probability of a swing, measured by the absolute Penrose-Banzhaf (PB) power index (Penrose 1946, Banzhaf 1965):

$$\Pi_i^{PB}(N, q, \mathbf{w}) = \frac{s_i}{2^{n-1}}$$

(s_i is the number of swings of the member i and 2^{n-1} is the number of configurations with i).

To compare the relative power of different committee members, a relative form of the PB power index is used:

$$\pi_i^{PB}(N, q, \mathbf{w}) = \frac{s_i}{\sum\limits_{k \in N} s_k}$$

Assuming many voting acts and all possible preference orderings are equally likely, it makes sense to evaluate an a priori voting power of each committee member as a probability of being in pivotal situation, measured by Shaply-Shubik (SS) power index (Shapley and Shubik 1954):

$$\Pi_i^{SS}(N, q, \mathbf{w}) = \frac{p_i}{n!}$$

(p_i is the number of pivotal positions of the committee member i, and n! is the number of permutations of all committee members). Since $\sum_{i \in N} p_i = n!$,it holds that

$$\pi_i^{PB}(N, q, \mathbf{w}) = \frac{s_i}{\sum\limits_{k \in N} s_k},$$

i.e. absolute and relative form of the SS-power index is the same.[2]

[2] Supporters of Penrose-Banzhaf power concept are sometimes refusing Shapley-Shubik index as a measure of voting power. Their objections to Shapley-Shubik power concept are based on classification of power measures on so called I-power (voter's potential influence over the outcome of voting) and P power (expected relative share in a fixed prize available to the winning group of committee members, based on cooperative game theory) introduced by Felsenthal, Machover and Zwicker (1998). Shapley-Shubik power index was declared to represent P-power and as such unusable for measuring influence in voting. We tried to show (Turnovec 2007, Turnovec, Mercik, Mazurkiewicz 2008) that objections against Shapley-Shubik power index, based on its interpretation as a P-power concept, are not sufficiently justified. Both Shapley-Shubik and Penrose-Banzhaf measure could be successfully derived as cooperative game values, and at the same time both of them can be interpreted as probabilities of being in some decisive position (pivot, swing) without using cooperative game theory at all.

Let us denote by $W[N, q, \mathbf{w}]$ the set of all winning configuration in a simple quota weighted committee $[N, q, \mathbf{w}]$ generated by voting rule q. By

$$\varepsilon(N, q, \mathbf{w}) = \frac{\# W[N, q, \mathbf{w}]}{2^n}$$

we denote the efficiency of voting rule q, which is the probability that a proposal will be passed in committee $[N, q, \mathbf{w}]$ providing all voting configurations (all preference orderings) are equally likely.[3]

It can be easily seen that for any $\alpha > 0$ and any power index it holds that $\Pi_i[N, \alpha q, \alpha\mathbf{w}] = \Pi_i[N, q, \mathbf{w}]$. Therefore, without loss of generality we can assume that $\sum_{i \in N} w_i = 1$ and $0 < q \leq 1$, using in analysis only relative weights and relative quotas.

The committee $[N, q, \mathbf{w}]$ has a property of strictly proportional power (Berg and Holler 1986) if $\pi[N, q, \mathbf{w}] = \mathbf{w}$ (i.e. the relative voting power of committee members is equal to their relative voting weights). The case of strictly proportional power seldom occurs.

3. Fairness and Efficiency of the Voting System and Quota Intervals of Stable Power

A fair voting system is usually formulated in terms of voting weights. Let $\mathbf{w} = (w_1, w_2, ..., w_n)$ be a fair distribution of voting weights (whatever principle is used to justify it), then the voting system used is fair if the committee $[N, q, \mathbf{w}]$ has the property of strictly proportional power. For given N and $\mathbf{w}$ the only variable we can vary to design fair voting system is quota q.

Proposition 1 Let $[N, q, \mathbf{w}]$ be a simple quota weighted committee with a quota q,

$$\min_{S \in W[N,q,\mathbf{w}]} \left(\sum_{j \in S} w_j - q \right) = \mu^+(q) \geq 0$$

and

$$\min_{S \in 2^N \setminus W[N,q,\mathbf{w}]} \left(q - \sum_{j \in S} w_j \right) = \mu^-(q) \geq 0$$

[3] The term 'efficiency' is rather misleading as it is something like an ability of a voting body to make decision. The concept itself is based on Coleman's 'ability of a collectivity to act' (Coleman 1971). In the model of simple quota weighted committee any voting act is a choice of one of two alternatives: the voted proposal (change of status quo) against status quo. The change is approved if it is supported by members representing at least total weight q, while the status quo is maintained in the opposite case. Henceforth the status quo is implicitly considered to be less 'desirable' than its change. In fact Coleman's concept, used in recent literature under the label of 'efficiency' gives the probability of changing the status quo.

Then for any q we have $W[N,q,\mathbf{w}] = W[N,\gamma,\mathbf{w}]$ for all

$$\gamma \in (q - \mu^-(q), q + \mu^+(q)].$$

The intervals $(q - \mu^-(q), q + \mu^+(q)]$ is called an interval of stable power for quota q. Quota $\gamma^* \in (q - \mu^-(q), q + \mu^+(q)$ is called marginal quota for q if $\mu^+(\gamma^*) = 0$.

Proof If $S \in W[N,q,\mathbf{w}]$ and $q < \gamma \le q + \mu^+(q)$, then

$$0 \le \sum_{j\in S} w_j - q - \mu^+(q) \le \sum_{j\in S} w_j - \gamma \le \sum_{j\in S} w_j - q \Rightarrow S[N,\gamma,\mathbf{w}]$$

If $S \in W[N,q,\mathbf{w}]$ and $q > \gamma \ge q - \mu^-(q)$, then

$$0 \le \sum_{j\in S} w_j - q \le \sum_{j\in S} w_j - \gamma \le \sum_{w_j - q + \mu^+(q)} \Rightarrow S \in W[N,\gamma,\mathbf{w}]$$

Winning configurations S for quota q are winning also for any quota $\gamma \in (q - \mu^-(q), q + \mu^+(q)]$.

If $S \in 2^N \backslash W[N,\gamma,\mathbf{w}]$ and $q < \gamma \le q + \mu^+(q)$, then

$$0 > \sum_{j\in S} w_j - q \ge \sum_{j\in S} w_j - \gamma \ge \sum_{w_j - q - \mu^+(q)} \Rightarrow S \in 2^N \backslash W[N,\gamma,\mathbf{w}].$$

If $S \in 2^N \backslash W[N,\gamma,\mathbf{w}]$ and $q > \gamma \ge q - \mu^-(q)$, then

$$0 > \sum_{j\in S} w_j - q + \mu^-(q) \ge \sum_{j\in S} w_j - \gamma \ge \sum_{w_j - q} \Rightarrow S \in 2^N \backslash W[N,\gamma,\mathbf{w}].$$

Any losing configuration S for quota $\gamma \in (q - \mu^-(q), q + \mu^+(q)]$ is losing also for quota q. □

Example 1 Consider the committee $[N,q,\mathbf{w}]$ where $n = 3, w_1 = 0.2, w_2 = 0.2, w_3 = 0.6$. Then $2^N = (\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\})$. Set $q = 0.51$. Then

$$W[N,q,\mathbf{w}] = (\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}),\ 2^N \backslash W[N,q,\mathbf{w}] = (\varnothing, \{1\}, \{2\}, \{1,2\}).$$

$\mu^+(q) = \min\{w3 - 0.51 = 0.09, w_1 + w_3 - 0.51 = 0.29, w_2 + w_3 - 0.51 = 0.29, w_1 + w_2 + w_3 - 0.51 = 0.49\} = 0.09.$

$\mu^-(q) = \min\{0.51 - w_1 = 0.31, 0.51 - w_2 = 0.31, 0.51 - w_1 - w_2 = 0.11\} = 0.11.$

Thus the quota interval of stable power for $q = 0.51$ is $(0.4, 0.6]$, the marginal quota for $q = 0.51$ is $\gamma^* = 0.6$.

Now define a partition of the power set 2^N into equal weight classes Ω_0, Ω_1, …, Ω_r (such that the weight of different configurations from the same class is the same and the weights of different configurations from different classes are different). Clearly it holds that $r \leq 2^n - 1$. For completeness, set $w(\varnothing) = 0$. Consider weight increasing ordering of equal weight classes $\Omega^{(0)}, \Omega(1), \ldots, \Omega^{(r)}$ such that for any $t < k$ and $S \in \Omega(t)$, $R \in \Omega(k)$ it holds that $w(S) < w(R)$. Denote $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 1, 2, \ldots, r$.

Proposition 2 Let $\Omega^{(0)}, \Omega^{(1)}, \ldots, \Omega^{(r)}$ be the weight increasing ordering of equal weight partition of 2^N. Set $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 0, 1, 2, \ldots, r$. Then there is a finite number $r \leq 2^n - 1$ of marginal quotas q_t and corresponding intervals of stable power $(q_{t-1}, q_t]$ such that $W[N, q_t, \mathbf{w}] \subset W[N, q_{t-1}, \mathbf{w}]$.

Example 2 In the committee from Example 1, order all voting configuration by their weights:

t	$\Omega^{(t)}$	$w(S)$
0	$\varnothing$	0.0
1	$\{1\}, \{2\}$	0.2
2	$\{1, 2\}$	0.4
3	$\{3\}$	0.6
4	$\{1, 3\}, \{2, 3\}$	0.8
5	$\{1, 2, 3\}$	1.0

We have 6 classes of equal weight voting configurations ordered by weights and $r = 5$. There exist 5 quota intervals of stable power and corresponding marginal quotas and 5 different vectors of power indices:

t	Interval	Marginal quota	# WC	SS power
1	$(0, 0.2]$	0.2	7	$(1/3, 1/3, 1/3)$
2	$(0.2, 0.4]$	0.4	5	$(1/6, 1/6, 4/6)$
3	$(0.4, 0.6]$	0.6	4	$(0, 0, 1)$
4	$(0.6, 0.8]$	0.8	3	$(1/6, 1/6, 4/6)$
5	$(0.8, 1]$	1	1	$(1/3, 1/3, 1/3)$

Proposition 3 Let $q_1, q_2, \ldots, q_r$ be the set of all marginal quotas in simple quota weighted committee $[N, q, \mathbf{w}]$ and π^k be a vector of relative power indices corresponding to marginal quota q_k, then there exists a vector $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that

$$\sum_{k=1}^{r} \lambda_k = 1,\ \lambda_k \geq 0,\ \sum_{k=1}^{r} \lambda_k \pi^k = \mathbf{w}$$

Proof follows from Berg and Holler (1986), who introduced the concept of strictly proportional power. They provide the following property of simple weighted committees: Let $[N, Q, \mathbf{w}]$ be a finite family of simple quota weighted committees with the same weights $\mathbf{w}$ and set of different relative quotas $Q = \{q_1, q_2, \ldots, q_m\}$. Let$\varphi(Q)$ be a probability distribution over Q where φ_k is a probability with which a random mechanism selects the quota q_k and $\pi_{ik}(N, q_k, \mathbf{w})$ be a power index in the committee $[N, q_k, \mathbf{w}]$ with a quota $q_k \in Q$, then

$$\overline{\pi}_i(N, Q, \mathbf{w}) = \sum_{k: q_k \in Q} \pi_{ik}(N, q_k, \mathbf{w}) \varphi_k$$

is an expected relative power of the member i in the randomized committee $[N, \lambda(Q), \mathbf{w}]$. For any vector of relative weights there exist a finite set Q of relative quotas q_k such that $0,5 < q_k \leq 1$, and a probability distribution λ such that

$$\overline{\pi}_i(N, Q(\lambda), \mathbf{w}) = \sum_{q_k \in Q} \pi_{ik}(N, q_k, \mathbf{w}) \lambda_k = w_i$$

Randomized voting rule $\lambda(Q)$ leads to strictly proportional power. □

Example 3 Randomized voting rule in committee from Example 1 applied to Shapley-Shubik index:

$$\begin{aligned}
\frac{1}{6}\lambda_2 + \frac{1}{3}\lambda_3 &= \frac{1}{5} \\
\frac{1}{6}\lambda_2 + \frac{1}{3}\lambda_3 &= \frac{1}{5} \\
\lambda_1 + \frac{4}{6}\lambda_2 + \frac{1}{3}\lambda_3 &= \frac{3}{5} \\
\lambda_1 + \lambda_2 + \lambda_3 &= 1 \\
\lambda_j &\geq 0
\end{aligned}$$

The system has a unique solution:

$$\lambda_1 = \frac{1}{5}, \lambda_2 = \frac{2}{5}, \lambda_3 = \frac{2}{5}$$

If there is a random mechanism selecting marginal quotas $q_1 = 0.6, q_2 = 0.8, q_3 = 1$ with probabilities $\lambda_1 = \frac{1}{5}, \lambda_2 = \frac{2}{5}, \lambda_3 = \frac{2}{5}$, then mathematical expectation of SS-power of the members of the committee will be equal to their relative weights (we obtain the case of strictly proportional power).

One can hardly expect that randomized voting rules leading to strictly proportional power would be adopted by actors of real voting systems. However, design of a 'fair' voting system can be based on an approximation provided by quota generating minimal distance between vectors of power indices and weights.

In political science, the concept of deviation from proportionality is used defined as follows: Let v_i be a share of votes political party i obtained in election and s_i be a share of seats allocated to party i in the elected body, then deviation from proportionality index is defined as:

$$\delta(\mathbf{s}, \mathbf{v}) = 1 - d(\mathbf{s}, \mathbf{v})$$

where $d(s, w)$ is a normalized distance between vectors **s** and **w** (with values between 0 and 1). Clearly $0 \leq \delta(s, v) \leq 1$, $\delta(s, v) = 1$ means full proportionality, $\delta(s, v) = 0$ means full disproportionality. Depending on used definition of distance political science proposes Loosemore-Hanby (1971) absolute values deviation metric

$$d_{LH}(\mathbf{s}, \mathbf{v}) = \frac{1}{2} \sum_i |(s_i - v_i)|,$$

that leads to proportionality index

$$\delta_{AV}(\mathbf{s}, \mathbf{v}) = 1 - \frac{1}{2} \sum_i |(s_i - v_i)|,$$

or least squares metric (Gallagher 1991)

$$d_{LS}(\mathbf{s}, \mathbf{v}) = \sqrt{\frac{1}{2} \sum_i (s_i - v_i)^2},$$

that leads to least squares proportionality index

$$\delta_{LS}(\mathbf{s}, \mathbf{v}) = 1 - \sqrt{\frac{1}{2} \sum_i (s_i - v_i)^2}.$$

By analogy let us introduce least squares index of fairness substituting relative weights w_i for v_i and relative power π_i for s_i:

$$\phi_{LS}(\pi[N,q,\mathbf{w}],\mathbf{w}) = 1 - \sqrt{\frac{1}{2}\sum_i (\pi_i[N,q,\mathbf{w}] - w_i)^2}$$

Considering ϕ to be a function of q, a good approximation of a fair quota is a marginal quota maximizing index of fairness.

To maximize ϕ is the same as to minimize sum of square residuals between the power indices and voting weights by q:

$$\sigma^2(q) = \sum_{i\in N} (\pi_i(N,q,\mathbf{w}) - w_i)^2.$$

Subject to condition $\frac{1}{2} < q \leq 1$. The quota minimizing σ^2 was introduced by Słomczyński and Życzkowski (2006, 2007) and called an *optimal quota*.

Słomczyński and Życzkowski (2007: 393) introduced the optimal quota concept within the framework of the so-called Penrose voting system as a principle of fairness in the EU Council of Ministers voting measured by Penrose-Banzhaf power index. The system consists of two rules:

1. The voting weight attributed to each member of the voting body of size n is proportional to the square root of population he or she represents.
2. The decision of the voting body is taken if the sum of the weights of members supporting it is not less than the optimal quota.

Looking for a quota providing a priori voting power 'as close as possible' to the normalized voting weights, Słomczyński and Życzkowski are minimizing the sum of square residuals between the power indices and voting for $q \in (0.5, 1]$. They propose two heuristic approximations of the solution:

$$q_s = \frac{1}{2}\left(1 + \frac{1}{\sqrt{n}}\right)$$

and

$$q_n = \frac{1}{2}\left(1 + \sqrt{\sum_{i\in N} w_i^2}\right)$$

Proposition 4 Let $[N,q,\mathbf{w}]$ be a simple quota weighted committee, then there exists exact solution of Słomczy ński and Życzkowski optimal quota (SZ optimal quota) problem:

$$q^* = \arg\min_j \sum_{i\in N} \left(\Pi_i(N,q_j,\mathbf{w}) - w_i\right)^2$$

where $j = 1, 2, \ldots, r$ is the number of intervals of stable power such that q_j are marginal majority quotas (greater than $\frac{1}{2}$).

Proof follows from finite number of quota intervals of stable power (Proposition 2). Quota q^* provides best approximation of strictly proportional power, that is related neither to particular power measure nor to specific principle of fairness. □

Example 4 The index of fairness for majority marginal quotas in committees from Example 1 gives:

Member	Weight	SS $q_3 = 0.6$	SS $q_4 = 0.8$	SS $q_5 = 1.0$
1	0.2	0.000	0.167	0.333
2	0.2	0.000	0.167	0.333
3	0.6	1.000	0.667	0.333
$\sum$	1.0	1.000	1.000	1.000
$\phi(\mathrm{q}_t)$		0.654	0.942	0.769

The exact optimal quota is $q* = q_4 = 0,8$ (and all quotas $q \in (0.6, 0.8])$. Compare to Słomczyński and Życzkowski approximations: $q_s = 0.79, q_n = 0.83$.

Together with problem of legitimacy (fairness) designers of voting systems are concerned with ability of voting body to change status quo (efficiency),

Proposition 5 Let $[N, q, \mathbf{w}]$ be a simple weighted committee, then the efficiency index

$$\varepsilon(N, q, \mathbf{w}) = \frac{card\ W[N, q, \mathbf{w}]}{2^n}$$

is non-increasing function of quota q and attains finite number of values between 0 and 1.

Proof Follows from properties of marginal quotas: finiteness of the number of marginal quotas and strict inclusiveness $W(N, \gamma^{t+1}, \mathbf{w}) \subset W(N, \gamma^t, \mathbf{w})$, see Proposition 2. □

Being able to calculate all marginal quotas we have all possible levels of efficiency in simple weighted committee and can compare them with appropriate values of index of fairness for majority quotas. This information is provided by fairness-efficiency matrix:

$$\begin{pmatrix} \sigma^2(N, q_1, \mathbf{w}) & \varepsilon(N, q_1, \mathbf{w}) \\ \vdots & \\ \sigma^2(N, q_r, \mathbf{w}) & \varepsilon(N, q_r, \mathbf{w}) \end{pmatrix}$$

where $q_1, q_2, \ldots, q_r$ are majority marginal quotas (such that $\frac{1}{2} < q_1 < q_2 < \ldots < q_r \leq 1$); the rows correspond to marginal quotas and columns to fairness index and efficiency index.

While the index of fairness is not a monotonic function of the quota, the efficiency index is strictly decreasing with an increase of the quota. Thus there is always a problem of trading off between fairness and efficiency: a choice of a row from the fairness-efficiency matrix, using approaches of multi-criteria optimization.

Example 5 Take the fairness-efficiency matrix for marginal majority quotas in the committee from Example 1.

Majority marginal quota	# WC	ε	ϕ
0.6	4	0.5	0.65359
0.8	3	0.375	0.942265
1.0	1	0.125	0.76906

4. Concluding remarks

The fairness, efficiency, and approximation of strictly proportional power in voting systems is not exclusively related to the Penrose square-root rule and Penrose-Banzhaf definition of power, as it is usually done in discussions about the EU voting rules. In this paper it is treated in a more general setting as a property of any simple weighted committee and any well defined power measure. The choice of a 'fairness principle' in the EU decision making is a problem of political consensus of member states and cannot be resolved by 'scientific community' and by mathematical models, but the clarification and clear formulation and representation of the problem can be of help in political decisions. What one can expect from public choice theory is a contribution to mathematically rigorous implementation of a selected principle.

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