Problem #5 from Baltagi, Chapter 13

Truncated Uniform Density. Let x be a uniformly distributed random variable with density
f(x) =1/2 for $-1 < x < 1$.

(a) What is the density function of *f (x/x* > -1/2)? Hint: Use the definition of a conditional density f (x/x > -1/2) = f(x)/Pr[x > -1/2] for $-1/2 < x < 1$.

Solution: f (x/x > -1/2) = f(x) / Pr[x > -1/2]

P[x>-1/2] = 1 - F(-1/2) = 1 - 1/4 = 3/4

f(x|x>-1/2) = (1/2) / (3/4) = 2/3 for $-1/2 < x < 1$.

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-- (b)

(b) What is the conditional mean E(x/x > -1/2)? How does it compare with the unconditional mean of x? Note that because we truncated the density from below, the new mean should shift to the right.

Solution: $\mathrm{E}(x \mid x > -1/2) = \int_{-1/2}^{1} x \cdot f(x)\, dx / \mathrm{P}[x > -1/2]$

$$= \frac{\int_{-1/2}^{1} x \cdot 1/2\, dx}{1 - F(-1/2)} = 4/3 \cdot \left[1/4 \cdot x^2\right]_{-1/2}^{1} = 4/3 \cdot [1/4 - 1/16]$$

$$= 4/3 \cdot 3/16 = 1/4$$

-- (c)

(c) What is the conditional variance var(x/x > -1/2)? How does it compare to the unconditional var(x)? (Truncation reduces the variance).

Solution: $\text{var}(x \mid x > -1/2) = \text{E}\left(x^2 \mid x > -1/2\right) - [\text{E}(x \mid x > -1/2)]^2$

$= \text{E}\left(x^2 \mid x > -1/2\right) - (1/4)^2 = **$

$\text{E}\left(x^2 \mid x > -1/2\right) = \left(\int_{-1/2}^{1} f(x) \cdot x^2 \, dx\right) \Big/ \, (\text{P}[x > -1/2])$

$= \left(\int_{-1/2}^{1} 1/2 \cdot x^2 \, dx\right) \Big/ \, (1 - F(-1/2)) = \frac{4}{3} \cdot \left[\frac{1}{6} - \left(-\frac{1}{48}\right)\right]$

$= \frac{4}{3} \cdot \left(\frac{1}{6} + \frac{1}{48}\right) = \frac{4}{3} \cdot \left(\frac{8+1}{48}\right) = \frac{3}{12} = \frac{1}{4}$

$** = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$

Problem ♯5 from Baltagi, Chapter 13

Truncated Normal Density. Let x be N(1, 1). Using the results in the Appendix, show that:

(a) The conditional density f(x/x > 1) = 2φ(x -1) for x > 1 and f(x/x < 1) = 2φ(x -1) for x < 1.

Notation:

$\phi(.)$ probability density function (PDF) of N(0, 1)

$\Phi(.)$ cummulative distribution function (CDF) of N(0, 1)

$N(\mu, \sigma^2)$: $f(y \mid \mu, \sigma^2) = \frac{1}{\sigma} \cdot \phi\left(\frac{y-\mu}{\sigma}\right)$

$F(y \mid \mu, \sigma^2) = \Phi\left(\frac{y-\mu}{\sigma}\right)$

$P(y > c) = 1 - \Phi\left(\frac{c-\mu}{\sigma}\right)$

Solution: i) $f(x \mid x > 1), \ \mu = 1, \ \sigma = 1, \ c = 1$

$$f(x \mid x > 1) = \frac{f(x)}{P[x>1]} = \frac{1/\sigma \cdot \phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} = \frac{\frac{1}{1} \cdot \phi\left(\frac{x-1}{1}\right)}{1-\Phi\left(\frac{1-1}{1}\right)} = \frac{\phi(x-1)}{1/2} = 2 \cdot \phi(x-1)$$

ii) $f(x \mid x < 1), \ \mu = 1, \ \sigma = 1, \ c = 1$

$$f(x \mid x < 1) = \frac{f(x)}{P[x<1]} = \frac{1/\sigma \cdot \phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{c-\mu}{\sigma}\right)} = \frac{\frac{1}{1} \cdot \phi\left(\frac{x-1}{1}\right)}{\Phi\left(\frac{1-1}{1}\right)} = \frac{\phi(x-1)}{1/2} = 2 \cdot \phi(x-1)$$

-- (b)

(b) The conditional mean E(x/x > 1) = 1 + 2φ(0) and E(x/x < 1) = 1 -2φ(0). Compare with the unconditional mean of x.

Solution: i) $\mathrm{E}(x \mid x > 1)$, $\mu = 1$, $\sigma = 1$, $c = 1$

$$\mathrm{E}(x \mid x > 1) = \mu + \sigma \cdot \frac{\phi\left(\frac{\mu - c}{\sigma}\right)}{\Phi\left(\frac{\mu - c}{\sigma}\right)} = 1 + 1 \cdot \frac{\phi\left(\frac{1-1}{1}\right)}{\Phi\left(\frac{1-1}{1}\right)} = 1 + \frac{\phi(0)}{1/2} = 1 + 2 \cdot \phi(0)$$

ii) $\mathrm{E}(x \mid x < 1)$, $\mu = 1$, $\sigma = 1$, $c = 1$

$$\mathrm{E}(x \mid x < 1) = \mu - \sigma \cdot \frac{\phi\left(\frac{c - \mu}{\sigma}\right)}{\Phi\left(\frac{c - \mu}{\sigma}\right)} = 1 - 1 \cdot \frac{\phi\left(\frac{1-1}{1}\right)}{\Phi\left(\frac{1-1}{1}\right)} = 1 - \frac{\phi(0)}{1/2} = 1 - 2 \cdot \phi(0)$$

-- (c)

(c) The conditional variance var(x/x > 1) = var(x/x < 1) = 1 -4φ2(0). Compare with the unconditional variance of x.

$$\text{var}(x \mid x > c) = \sigma^2 \left\{1 - \frac{\sigma\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} \cdot \left[\frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{1-\Phi\left(\frac{c-\mu}{\sigma}\right)} - \left(\frac{c-\mu}{\sigma}\right)\right]\right\}$$

$$\text{var}(x \mid x > c) = \sigma^2 \left\{1 + \frac{\sigma\left(\frac{c-\mu}{\sigma}\right)}{\Phi\left(\frac{c-\mu}{\sigma}\right)} \cdot \left[-\frac{\phi\left(\frac{c-\mu}{\sigma}\right)}{\Phi\left(\frac{c-\mu}{\sigma}\right)} - \left(\frac{c-\mu}{\sigma}\right)\right]\right\}$$

$$\text{P}(y > c) = 1 - \Phi\left(\frac{c-\mu}{\sigma}\right)$$

Solution: i) $\text{var}(x \mid x > 1),\ \mu = 1,\ \sigma = 1,\ c = 1$

$$\text{var}(x \mid x > 1) = 1 \cdot \left\{1 - \frac{\sigma(0)}{1/2} \cdot \left[\frac{\phi(0)}{1/2} - 0\right]\right\} = 1 - 4 \cdot \phi^2(0)$$

ii) $\text{var}(x \mid x < 1),\ \mu = 1,\ \sigma = 1,\ c = 1$

$$\text{var}(x \mid x > 1) = 1 \cdot \left\{1 + \frac{\sigma(0)}{1/2} \cdot \left[-\frac{\phi(0)}{1/2} - 0\right]\right\} = 1 - 4 \cdot \phi^2(0)$$