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Electoral competition for the 2+1 electoral rule and the close alternatives

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Electoral competition for the 2+1 electoral rule and the close alternatives

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Abstract:
The Anglo-American double-member districts employing plurality-at-large are frequently criticized for giving a large majority premium to a winning party, since the large premium may decrease proportionality of the elected assembly relative to single-member districts. We demonstrate that the premium stems from a limited degree of voters' discrimination associated with only two positive votes on the ballot. To enhance voters' ability to discriminate, we consider alternative electoral rules that give voters more positive and negative votes. We identify strict voting equilibria of several alternative rules in a situation where candidates differ in binary ideology and binary quality, voters' ideology-types are binomially distributed, voters are strategic, and a candidate's policy is more salient than candidate's quality. The most generous rules such as approval voting and combined approval-disapproval voting only replicate the electoral outcomes of plurality-at-large. The best performance in a double-member district is achieved by a rule that assigns two positive votes and one negative vote to each voter (2+1 rule). Under a strict and sincere pure-strategy equilibrium of the 2+1 rule, the second largest group frequently wins the second seat and high-quality candidates gain seats more likely

than low-quality candidates. The 2+1 rule increases the scope for a voter's discrimination while avoiding the underdog effects and overstating of preferences associated with an unrestricted number of negative votes.

Keywords: electoral rules, strategic voting, negative votes, plurality-at-large

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1 Introduction

In the year 2012, a Czech financier, mathematician and philanthropist Karel Janeček, proposed a '2+1 electoral rule'.[1] This scoring electoral rule for double-member districts gives each voter two positive votes and a single negative vote, where the positive and negative points add up to each other. A voter cannot cumulate positive points to a single candidate, but partial abstention is allowed. A unique feature of the 2+1 electoral rule is the simultaneous presence of the positive and negative votes, hence the rule blends structurally different properties of best-rewarding and worst-punishing electoral rules. Motivation for the rule is to allow voters to safely express their policy preferences but at the same time motivate them for discrimination along the quality (i.e., competence or integrity) dimension.

This paper attempts to compare properties of the 2+1 rule relative to the properties of closely similar rules in two-member districts, including plurality-at-large that is currently almost exclusively used in Anglo-American double-member districts. Experience from nine U.S. states demonstrates that plurality-at-large in two-member districts is associated with a low incidence of split outcomes (Cox, 1984). In fact, in the most recent elections of 2010–2012 in the U.S. states, a single party gained both seats in 78.5% of two-member districts. The large premium for a majority party in a district is seen as the major disadvantage of having the two-member districts as such, and district magnitude has even become a constitutional issue in the United States. In a famous 1986 decision, Thornburg v. Gingles, the U.S. Supreme Court overturned North Carolinas multi-member legislative lines on the grounds that they discriminated against blacks. The U.S. Voting Rights Act thus encourages the creation of districts where racial or ethnic minorities predominate, and single-member districts are interpreted as best fitting this objective.

Our analysis of the 2+1 electoral rule and close alternatives illustrates that the shortcomings of plurality-at-large are not necessarily overcome by adopting single-member districts with plurality rule. We argue that disproportionality in two-member districts may be rather addressed by reforming the structure of the ballot towards endowing voters with more than just two positive votes, and demonstrate that the resulting outcomes improve not only representativeness, but also quality of the elected representatives. For that purpose we investigate several scoring rules for two-member districts, including plurality-at-large, 2+1 rule, approval voting, and the most generous combined approval-disapproval voting. We confine the analysis to an elementary electoral situation with exactly one valence dimension and one policy dimension, which contains the essential tradeoff between policy and quality as faced by instrumentally rational voters.

We employ the standard calculus of voting for multi-candidate elections (McKelvey and Ordeshook, 1972; Palfrey, 1989; Myerson and Weber, 1993; Myerson, 1993; Cox, 1994). Our

[1] Source: `http://www.kareljanecek.com/muj-navrh-volebniho-zakona` (in Czech), Accessed 16 November, 2012

model belongs into a class of voting models with population uncertainty, where the set of voters is driven from a probability distribution. Pivotal events where an individual vote can make a difference are essential for characterizing the optimal ballots. To introduce population uncertainty into two-member districts, we draw upon a multinomial model with a predetermined total number of voters (Palfrey, 1989; Cox, 1994; Carmona, 2012). Our binomial distribution of voters and simple preferences allow us to derive a few easily interpretable insights on the effects of alternative scoring electoral rules in two-member districts that serve as a point of departure for investigations of more complex electoral situations.

In voting theory with population uncertainty, most of attention has been devoted to the analysis of scoring rules in single-member districts (Myerson, 2002; Myatt, 2007; Krishna, Morgan, 2011; Bouton, Castanheira, 2012). While certain results and methods can be swiftly translated to multi-member districts, the main difference is that decisive races in multi-member districts are contests for the last seat, not for the first seat, thus pivotal events have a more complicated structure than in single-member districts. A low expected score of a candidate need not indicate that the candidate is now serious in the main pivotal events and vice versa. For example, Bouton and Castanheira (2012) make a difference between restricted and unrestricted magnitudes of the events for single-member districts, and compute the unrestricted magnitudes, while leaving restricted magnitudes only bounded. But for the two-member districts, all relevant magnitudes are restricted magnitudes.

We construct a model of two ex ante symmetric groups of voters (lef-wing L-voters and right-wing R-voters), with a predetermined total size. The relative size of the groups is being random, with each individual type drawn independently and identically. Like in Myerson (1993), each ideological type is represented by a single low-quality and a single high-quality candidate. Thus, four generic types of candidates, differing in binary ideology and binary quality, run for seats. We identify all relevant voting equilibria for selected electoral rules, and focus primarily upon the strict equilibria. The rules differ in the number of positive votes ($V^+ = 0, 1, 2, 3$) and the number of negative votes ($V^- = 0, 1, 2, 3$).

In the first step, we demonstrate the differences between the equilibrium voting profiles in single-member and two-member districts in our electoral situation.[2] By Duverger Law, for *plurality* rule (referred to as 1+0 rule), the set of seriously competing candidates is restricted to a pair of candidates, hence to a single binary dimension which may be a pure ideology dimension. There are two adverse effects: First, adherents of the same ideological platform need to coordinate on a preferred candidate. Second, valence dimension may be entirely suppressed in the binary competition.

Adding a negative vote (1+1 rule) does not robustly solve the problem, since at pivotal

[2]The comparison rests on the idea that quality competition is present both in single-member and two-member districts, hence four candidates run for office in both districts. If each group is represented by a single party and each party nominates a single candidate, then valence competition is entirely absent in the elections and the electoral outcome in single-member district is invariant to the rule.

events, the negative votes of one group tend to fully cancel out positive votes of the other group. Consequently, a large set of serious candidates among which the voters have to spread their few votes is generated at the pivotal events. The 1+1 rule contains a well-known 'underdog' effect of pure negative voting (c.f. Myerson, 1999) since *the rule embeds too many negative votes relative to positive votes.* In contrast, adding extra positive votes to simple plurality rule (approval voting) helps, since casting a positive vote to the high-quality candidate from own group poses zero strategic risk. As argued elsewhere (Myerson, 1993), approval voting is fully effective in promoting quality in single-member districts.

In two-member districts, the number of viable candidates may grow by Duvergerian hypotheses, hence coordination failure is less likely. However, once ideology dimension dominates quality dimension, *plurality-at-large* (2+0 rule) motivates the voters to support two preferred ideological candidates independently on their qualities. As a result, the dominant voting group always elects their two candidates independently on quality. What if we add extra positive votes? Approval voting does not improve the outcome except for some extreme parametrical realizations. Endowing the voter with an arbitrary number of both positive and negative votes (combined approval-disapproval rule) does not help either.

Adding only a single negative vote (2+1 rule) now generates the following incentives: Each group of voters tries to win the two seats for their two candidates. In the electoral competition, two types of pivotal events arise: (i) if the group is in majority (majority event), the stronger candidate of the group gains the first seat and the weaker candidate of the group competes with the stronger candidate of the opposing group for the second seat. (ii) If the group is in minority (minority event), the group cannot win the first seat and its stronger candidate competes with the weaker candidate of the other group for the second seat. Voters rationally cast two positive votes to two candidates from their group to win both types of pivotal events; qualities of candidates in the two events do not matter for the optimal allocation of positive votes. In contrast, quality matters for how the single negative vote is allocated, because the single negative vote is cast such that only a single (more valuable) type of pivotal events is influenced.

We identify existence conditions for a strict and sincere pure-strategy equilibrium. In a sincere equilibrium, the negative vote is cast to low-quality candidate of the other group, and the low-quality candidates become weak candidates. Outcomes generated by the sincere profile under the 2+1 rule improve upon the majority-driven outcomes of plurality-at-large rule both in quality and representativeness. The fact that negative votes are cast to the low-quality candidate from the opponent group increases the average quality of the elected candidates and protects minority interests more than alternative rules in two-member districts, including approval voting. Additionally, both positive votes and negative vote are cast sincerely, hence honest voters behave identically as sophisticated voters.

The sincere equilibrium exists if and only if *the minority event is more valuable under a sincere profile.* There are two effects that shape this existence condition. The first effect is associated with win gains in the two events. In the minority event, *both* policy and quality of the elected representative are in stake: The strong and high-quality candidate from own group competes with the weak and low-quality candidate from the other group. In the majority event, policy is at stake, yet the quality is worsened in the case of victory. The difference of the win gains contributes to minority events being more valuable. The second effect is related to different posteriors of the groups about the distribution of voters. Based on a private signal of own type, each voter has private beliefs about the distribution of types in population. Relative to an external observer, each voter is optimistic in a sense of expecting the majority event as more likely and the minority event as less likely. This optimism weakens the incentive to cast the negative vote in order to switch the minority events.

For the sincere equilibrium, the former effect must dominate the latter. This is not guaranteed in largely asymmetric districts, where the second effect exceeds the first effect at least for the group that has ex ante larger probability of being in majority (an ex-ante-majority group). In equilibrium of such an asymmetric district, the ex-ante-minority group remains sincere, and the ex-ante-majority group strategically mixes her negative vote, with a larger probability of the negative vote cast to the low-quality candidate of the ex-ante-minority group. As a result, the ex-ante-majority group increases the number of viable candidates of the ex-ante-minority group. In any case, even if some voters strategically mix partly against the high-quality candidate of the other group, the electoral outcome under the 2+1 rule still quality-dominates the outcome under plurality-at-large where low-quality candidates and high-quality candidates are treated identically.

Section 2 motivates the key assumptions. Section 3 builds the setup and discusses general properties of the relevant voting equilibria. Section 4 analyzes three electoral rules for single-seat districts (plurality, mixed voting and approval voting). Section 5 analyzes four electoral rules for two-seat districts (plurality-at-large, 2+1 rule, approval voting, and combined approval-disapproval voting). Section 6 offers extensions. Section 7 concludes.

2 Building blocks

To begin with, we motivates four key elements of the paper: competition in an exogenous valence dimension, a small set of generic candidates, the set of scoring rules, and the structure of population uncertainty.

2.1 Valence competition

In this paper, we conduct a elementary analysis of the electoral competition where we let candidates for office compete both in a conflicting (policy) dimension but also in a non-conflicting (quality or valence) dimension.[3] Candidate's quality can be also interpreted as

[3]In its broad definition, valence is used for any valuable personal characteristic including campaigning or networking skills, but in a narrow sense, valence is only for the qualities that voters value for their own

corruptability, hence valence competition represents one of many channels between electoral rules and corruption (Persson et al., 2003).[4]

We assume that both policy preferences (ideological biases) and valence are predetermined and known. The reason is to isolate the pure and direct effect of the electoral rule on the electoral expectations about the high- and low-quality candidates, and thereby on the calculus of voting. We leave the analysis of the interaction of the electoral rules and strategic position-taking into extensions. When parties endogenously determine the set of competing candidates, the literature agrees that low-quality candidates adopt extreme positions to soften valence competition, while high-quality candidates adopt centrist positions (Groseclose, 2001; Aragones, Palfrey, 2002; Hollard, Rossignol, 2008; Hummel, 2010), even if valence is endogenous to campaigning (Carrillo, Castanheira, 2008; Ashworth, de Mesquita, 2009).

We presume that *only two dimensions* emerge as relevant dimensions describing the candidates. Under such assumption, the non-conflicting valence dimension can end up either as (i) irrelevant dimension (i.e., there is no tradeoff between quality and policy), (ii) relevant and strong dimension (i.e., low-quality candidates will be likely ousted by high-quality candidates), or (iii) relevant but weak dimension. Only the last option is non-trivial and will be explicitly considered.

An intriguing question is why high-quality candidates may become electorally weaker than low-quality candidates. The explanation lies in the determinants of seriousness of candidates. (By *seriousness* of a candidate, we mean that an individual vote for this candidate will likely affect the electoral result. A strategic voter cast votes primarily to contenders in close races, hence to the most serious candidates.) We observe two problems with determinants of seriousness:

First, the *strength* of a candidate, namely the candidate's probability of being elected, need not coincide with seriousness. Hence, even if high-quality candidates are seen as stronger candidates, they may not be more serious. For example, we will illustrate that in the symmetric strict equilibrium for the 2+1 rule, the most serious candidates for any voter are the most

sake such as integrity, competence, and dedication to public service. More spefically, Stone and Simas (2010) measure character-valence through seven indicators: personal integrity, ability to work well with other leaders, ability to find solutions to problems, competence, grasp of the issues, qualifications to hold public office, and overall strength as a public servant.

[4]Many effects of the electoral system upon the level of corruption are related to incentives and disincentives to raise rents depending on the size of the party system and coalitional behavior. These effects can be modeled independently on the qualities of the candidates or parties. Persson and Tabellini (2001) identify two effects: The first effect of multi-party systems is in the coalitional bargaining stage. A party with ideological similarity to the proposer becomes cheaper to include into a coalition than an ideologically distant party hence tends to claims large rents in the bargaining. The second effect is of diluted individual performance. For large coalitions and closed party-lists, misconduct of individual incumbents is more difficult to monitor, detect and punish in elections. In addition, there is a lower incentive of challengers to monitor and reveal incumbent's underperformance in multi-party systems, since revelation activity by one challenger generates an uncompensated positive externality to the other challengers (Charron, 2011).

extreme candidates (i.e., the best candidate and the worst candidate). At the same time, the best candidates are the strongest and the worst candidates are the weakest. This is different to single-member districts where seriousness of candidates coincides with strength of candidates. The most extreme case are strict equilibria in the deterministic setting without population uncertainty (c.f., Dellis 2013), where the two terms are equivalent.

The second problem is that even if seriousness may coincide with strength, the quality not necessarily coincides with strength. Voters may be trapped in a coordination failure where low-quality candidates are seen as strong and serious, and with positive voting, this makes them receive positive votes, hence become strong and serious. Simply, seriousness is an outcome of self-fulfilling expectations, and the expectations are not necessarily driven by quality of the candidate. To sum up, we cannot easily argue for a direct chain quality-strength-seriousness.

Even more specifically, in this paper, we explain that high valence of a candidate does not translate into an electoral advantage in the following cases: (i) There is a coordination failure that implies a complete lack of seriousness of the high-quality candidate. This is the classic version of the electoral 'barrier to entry'. (ii) The close or decidable races that involve only quality are relatively less valuable than close races that involve ideology, and therefore the voter has no incentive to discriminate between the candidates on the basis of quality. (iii) There is no close race that would involve only quality; any close race with a valence dimension involves also a conflicting policy issue. If the conflicting issue is a more important dimension, the voters discriminate along the policy dimension, not along the valence dimension.

2.2 Candidates

In our electoral situation, we assume competition of two ideological groups and exactly four candidates. Isn't the candidates' set K overly restricted for two-member districts? Motivation for having only a few candidates draws from the large Duvergerian literature on the number of serious candidates as an increasing function of the district magnitude M. The idea is that small district magnitudes make some social divisions latent and not expressed electorally.

Most of the vast Duvergerian literature examines a two-party prediction for simple plurality, called Duverger's Law. The early literature offers models with Duvergerian equilibria in which all votes for the second challenger vanish (Palfrey 1989, Myerson and Weber 1993, Cox 1994). Iaryczower and Mattozzi (2013) replicate Duverger's Law in the presence of campaigning efforts. Dellis (2013) confirms Duverger Law for any top-scoring rule in a deterministic setting and risk aversion, where a top-scoring rule (a rule with a unique top score) is defined such that it allows the voter to cast a different score for the first and second candidate on the ballot. An exception is Patty et al. (2009) who point to the presence of "too many" electoral equilibria in multi-candidate elections if the candidates are purely vote-seeking and adopt policy positions strategically. Among the most recent papers, Fujiwara (2011) uses a regression discontinuity in Brazilian mayoral elections to show that third-place candidates are more likely to be deserted in races under simple plurality rule than in runoff elections.

In the analysis of multiple-seat districts, Cox (1994; 1997) predicts $M + 1$ viable parties, and in the long run, $M + 1$ competing parties. This $M + 1$ result echoes results from all-pay auctions, where the number of contenders for winner-take-all contest with M prizes is typically $M + 1$ (Siegel, 2009). The use of $M + 1$ rule in entry predictions is conditional on 'precise expectations about prospective candidates' vote shares at the time entry decisions are made' (Cox 1999, p. 152). Yet Cox also warns that entry and viability are two different concepts in empirics, and argues that for $M > 5$, strategic voting is not possible because voters do not have good enough expectations about how well each party or candidate is likely to do in the upcoming election.

Morelli (2004) notes surprisingly little formal analysis on Duverger's hypotheses. The caveat is that Cox formally develops the $M + 1$ rule for single non-transferrable vote (a single positive vote for $M > 1$), and also his main evidence is primarily through district-level results from British and Japanese elections that used single non-transferrable vote. Jesse (1999) later confirms the results in two countries with the single transferable vote with multi-member districts. Under special assumptions, Duvergerian hypothesis can be however invalidated even on the district level such as in de Sinopoli and Iannantuoni (2007).

The lack of formal analysis of districts with larger magnitudes is understandable also because with larger magnitudes, there is a greater role for coordination on the nation-wide level and non-trivial linkages between district-wide and nation-wide competition (Cox, 1999). Morelli (2004) develops a model where for sufficiently asymmetric preferences across districts, the linkages revert Duvergerian predictions, hence the policy outcome with a proportional rule is more moderate than the one with plurality. Recent evidence (e.g., Singer and Stephenson 2009) is nevertheless supportive of the Duvergerian hypothesis, hence maintaining a low number of serious candidates is a reasonable point of departure.

2.3 Electoral rules in the Anglo-American two-member districts

Two-member districts were the norm in English elections from the thirteenth through most of the nineteenth century, until massive redistricting of 1885. The American political system inherited electoral laws from England, and the predominance of double- and other multi-member districts continued in the United States past the colonial period. The perspective of the time was more in favor of multi-member districts, where one of chief concerns with single-member districts was the excessive amount of special local legislation, as observed by New York constitutional commission in 1872 (Cox, 1984). Since the World War II, apportionments nonetheless led to a gradual adoption of single-member districts across the United States.[5]

[5] Another largely discussed example of the two-seat system with two positive votes is the 'binomial' open-list-PR system in Chile. In Chile, the system was designed the last years of the Pinochet regime with the aim to strategically overrepresent the coalition of the right-wing parties relative to center-left coalition. In the

In political science, double-member districts used to be seen as a prospective remedy to disproportionality in legislative representation associated with simple plurality: "*A two-member district PR system could achieve the functional purpose of plurality even better that the plurality method itself. And it would avoid some of the disadvantages of the plurality method such as ... its strong disincentive to the emergence of a major new party even when the electorate is dissatisfied with both parties in a two-party system.*" (Lijphart and Grofman, 1984, p. 8). Recently, Carey and Hix (2011) revised the tradeoff associated with the district magnitude, and for a sample of elections from 1945 to 2006 in all democratic countries with a population of more than one million, they find an optimal district magnitude to be in the range of three to eight.

Our interest is in performance of various electoral rules in two-member districts. Currently, there are 9 assemblies in the U.S. states that use two-member districts and elect both representatives per district through plurality-at-large. To understand the effect of plurality-at-large, we calculate the shares of mixed (split) and non-mixed (non-split) electoral outcomes, exploiting data from the most recent (2010–2012) elections.[6] In Table 1, the share of the mixed outcomes is approximately 21.4–21.5% of all outcomes, measured either as the average from all districts or the average of district averages. Put differently, in almost 80% of the electoral outcomes of plurality-at-large, a single party wins both seats.

Table 1: The shares of mixed electoral outcomes in two-member districts in the recent U.S. states elections

U.S. state	Election year	Assembly	Mixed	Total	Share
Arizona	2012	House of Representatives	2	28	7.1 %
Maryland	2010	House of Representatives	3	15	20 %
New Hampshire	2012	House of Representatives	17	53	32.1 %
New Jersey	2011	General Assembly	0	40	0 %
North Dakota	2012	House of Representatives	6	25	24 %
South Dakota	2012	House of Representatives	9	33	27.3 %
Vermont	2012	Senate	2	6	33.3 %
Vermont	2012	House of Representatives	14	45	31.1 %
West Virginia	2012	House of Delegates	2	11	18.2 %

The low incidence of mixed outcomes reported in Table 1 suggests that an increase in the

binomial system, if two parties are competing, the minority party gains a seat in the district (i.e., half of total seats) unless it obtains less than a third of votes. To achieve that level of representation was strategic also with respect to a two-third qualified majority requested for a constitutional change. Lacy and Niou (1998) provide an analysis relevant to strategic position-taking in the Chilean binomial system.

[6] Source: `http://ballotpedia.org/wiki/index.php/State_legislative_chambers_that_use_multi-member_districts`, Accessed 2 February, 2013

district magnitude from $M = 1$ to $M = 2$ may not improve representativeness. In theory, increasing the district magnitude improves representativeness of dispersed minorities at the expense of representativeness of concentrated minorities. In the context of the U.S. states, the negative effect upon concentrated minorities seems to outweigh the positive effect for dispersed minorities. The fact that two-member districts magnify disproportionality relative to single-member districts is seen as a major shortcoming of the two-member districts. Combined with a popular idea that single-member districts improve accountability, it is not surprising that reapportionments in many U.S. states are towards single-member districts.

Our paper aims to show that the source of disproportionality is primarily in the low degree of discrimination available with the plurality-at-large rule, not in the existence of the two-member districts. To understand the effect of electoral rules, we add extra positive and negative votes on the ballot. First, we add a single negative vote. Then we increase the number of positive votes to an arbitrary value (approval voting), and the number of both positive and negative votes to an arbitrary level (combined approval-disapproval voting, see Felsenthal, 1989). The combined approval-disapproval voting is the most generous electoral rule combining positive and negative votes. Notice that for $\#K \in \mathbb{N}$ delegates, the maximal strategically relevant number of votes is $\#K - 1$, hence approval voting is $(V^+, V^-) = (\#K - 1, 0)$ and combined approval-disapproval voting is $(V^+, V^-) = (\#K - 1, \#K - 1)$.

Our perspective is that voters need multiple degrees of discrimination to successfully incorporate secondary (here quality) considerations into their ballots. In the class of scoring rules, this requires simultaneous existence of both positive and negative votes. At the same time, there should not be too many negative votes to avoid the outcomes with too many serious candidates. Also, a cap to the use of multiple degrees of discrimination is needed. In the absence of the cap, voters tend to cast only extremal ballots to serious candidates in a sense that all serious candidates receive either a positive or a negative vote.[7]

Put in other words: To keep both degrees of discrimination (positive and negative) on the equilibrium ballot, we need to limit the number of positive and the number of negative votes to 'force' the voter to use also the intermediate values (in our case, zero points). At the same time, we do not want in principle to encourage many levels of discrimination such as in Borda Count simply to avoid strategic complexity in voting. The 2+1 rule is instrumental in setting the cap to the numbers of the votes and also in allowing to cast more positive votes than negative votes.

[7]This 'overstating' dominance holds almost always in a deterministic setting (Felsenthal, 1989). In a large (stochastic) voting game that is solved by Myerson-Weber's ordering condition, Nunez and Laslier (2013) find that a class of evaluative voting rule yields identical equilibria like approval voting. Under evaluative voting with $m \in \mathbb{N}$ points, a voter can assign up to m points to each candidate. Approval voting is a special case of evaluative voting where $m = 1$, and combined approval voting is a special case of $m = 2$.

2.4 Population uncertainty

We consider strategic voters, whose optimal ballot is determined by the structure of decisive events. The phenomenon of strategic or tactical voting has been identified in many contexts, including proportional and mixed systems, and ambiguity remains only over the size of the phenomenon. As an example, Kawai and Watanabe (2013) recently estimated a fully structural model of voting decisions in Japan's general election and concluded that between 63% and 85% of voters are strategic. Our intuition about the structure of pivotal events generated by various rules in two-member district elections is built around a stylized electoral situation, using by purpose the simplest tractable model of population uncertainty.

There are essentially two approaches to build population uncertainty and pivotal events in voting games: large Poisson games invented by Myerson (2000, 2002), and multinomial distributions of a finitely large electorate (Palfrey, 1989; Cox, 1994; Carmona 2012).[8]

A Poisson game is a unique population uncertainty game that satisfies independence of actions and environmental equivalence (Myerson, 1998). Large Poisson games have been applied to investigate selected scoring rules' properties for $M = 1$. Myatt (2007) demonstrates that in simple plurality, Duverger Law is driven by the assumption that voters know the distribution of party support in the constituency. Relaxing the assumption that the distribution of voter preferences is common knowledge, he shows that uncertainty about true party support results in negative feedback, which limits strategic voting. Nunez (2010) applies Poisson games to prove the lack of Condorcet consistency of approval voting. Krishna and Morgan (2011) study how endogenous turnout helps in the information aggregation function of scoring rules. Bouton and Castanheira (2012) demonstrate how approval voting may resolve both information aggregation problem and coordination problem of majorities.

In this paper, we adopt a model of binomial distribution of voters's types. In comparison to a large Poisson game, there are two main differences: With a fixed total number of voters n, the actions are not independent. Consider a pure strategy profile under simple plurality where each of two groups supports a different candidate. The event of k votes to one of the supported candidates is conditional on the event of $n - k$ votes for the other candidate. Second, the environments of individual types are not equivalent, and the model is generally not invariant to payoff-irrelevant subdivisions of types (no type-splitting invariance).

The lack of actions independence is not a major problem for the construction of pivotal events. Each voter knows that what matters is a small difference between votes for the candidates (close races). The small differences are characterized by particular shares of voting groups in the population, independently upon the size of the voting population. The only consequence of assuming a predetermined total number of voters is that the shares are realized in a unique event. Secondly, the lack of environmental equivalence can be directly incorpo-

[8] An alternative that builds pivotal events in the absence of population uncertainty is noise in recording votes (score uncertainty), where each strategy profile is associated with a distribution of various outcomes (Laslier, 2009).

rated into the analysis by considering each type's posterior belief. The lack of type-splitting invariance is addressed by selecting within-type-invariant strict equilibria in pure strategies.

More formally, suppose the numbers of L-voters and R-voters are two independent random variables $\hat{n}_L$ and $\hat{n}_R$, each with Poisson distribution with means n_L and n_R. The variable $\hat{n}_L + \hat{n}_R$ has also a Poisson distribution with mean $n_L + n_R$. Consider an individual L-voter. Given symmetry of the problem, we will be interested in her subjective probability of a tie characterized by a given $q := \frac{\hat{k}}{\hat{n}} \leq \frac{1}{2}$ share of L-voters and $1 - q = \frac{\hat{n}-\hat{k}}{\hat{n}}$ share of R-voters (call it Tie 1) relative to a subjective probability of a tie characterized by $1 - q = \frac{\hat{n}-\hat{k}}{\hat{n}}$ share of L-voters and $q = \frac{\hat{k}}{\hat{n}}$ share of R-voters (call it Tie 2), for any $\hat{n}$.

Consider any total number of voters $\hat{n} \in \mathbb{N}$ such that there exists $\hat{k} \in \mathbb{N} : \frac{\hat{k}}{\hat{n}} = \lfloor \frac{\hat{k}}{\hat{n}} \rfloor$. For L-voter, probability of Tie 1 with total $\hat{n}$ voters, $(\hat{n}_L, \hat{n}_R) = (\hat{k} - 1, \hat{n} - \hat{k})$, is

$$\Pr(\hat{k} - 1, \hat{n} - \hat{k}) = \frac{e^{-n_L} n_L^{\hat{k}-1}}{(\hat{k} - 1)!} \cdot \frac{e^{-n_R} n_R^{\hat{n}-\hat{k}}}{(\hat{n} - \hat{k})!}.$$

Probability of Tie 2 with total $\hat{n}$ voters, $(\hat{n}_L, \hat{n}_R) = (\hat{n} - \hat{k} - 1, \hat{k})$, is

$$\Pr(\hat{n} - \hat{k} - 1, \hat{k}) = \frac{e^{-n_L} n_L^{\hat{n}-\hat{k}-1}}{(\hat{n} - \hat{k} - 1)!} \cdot \frac{e^{-n_R} n_R^{\hat{k}}}{\hat{k}!}.$$

The ratio of the probabilities writes:

$$\frac{\Pr(\hat{k} - 1, \hat{n} - \hat{k})}{\Pr(\hat{n} - \hat{k} - 1, \hat{k})} = \frac{q}{1 - q} \cdot \left(\frac{n_L}{n_R}\right)^{2\hat{k}-\hat{n}}$$

Thus, when each population is driven from an identical population ($n_L = n_R$, ex ante symmetric district), the ratio of probabilities is invariant to $\hat{n}$. This is the assumption used in our benchmark model. With this ex-ante symmetry assumption, the analysis of pivotal events under a predetermined $\hat{n}$ is equivalent to the analysis of a finite (but not large) Poisson game. Once populations are driven from different populations, $n_L \neq n_R$ (asymmetric district), there are differences that we discuss in Section 6.2.

The general reason to avoid uncertainty over abstention is that with complicated ballots including both positive and negative votes, the pivotal events may have a very complex structure. The candidates' expected scores are not directly indicative of seriousness of the candidates, because serious candidates are the candidates who are in the tie over the second seat, not those candidates who are in the tie over the first seat. In our particular electoral situation with pure strategies, the differences in the structure of population uncertainty are fortunately relatively unimportant, since tie probabilities from the binomial setting with a fixed number of players can be closely approximated by the Poisson formula (Myerson, 1998). The direct advantage of the binomial setting is that the exact formulas for pivot probabilities are computationally more convenient.

3 The setup

3.1 Players

Consider two binary dimensions: policy is L or R, and quality is zero or one. Four generic types of candidates are admissible. We let the candidate list K involve exactly four candidates, one per each generic type, $K = \{L_1, L_0, R_1, R_0\}$.

Assume a large fixed number $n \in \mathbb{N}$ of voters. Purely for technical convenience, let $\lfloor \frac{n}{6} \rfloor = \frac{n}{6}$. Any voter is either of two types, $t = L, R$. Each voter's probability of being an L-type is drawn from an independent and identical Bernoulli distribution $p \in (0, 1)$. The number of L-voters is a random variable $k \in N$ on the support $N \equiv \{k \in \mathbb{N} : 0 \leqslant k \leqslant n\}$, with a binomial distribution $B(k; n; p)$.

Voters learn their private types right before the elections and do not communicate their types to the other voters. Voters makes an inference about the aggregate k by the posterior distribution functions, where the posterior probability distribution functions for L-type and R-types are $B_L(k; n; p)$ and $B_R(k; n; p)$. In the main analysis, we focus on the fully symmetric case, $p = \frac{1}{2}$, and simplify notation to $B(k)$. For $p = \frac{1}{2}$, $B(k) + B(n - k) = 1$, $B(\frac{n}{2}) = \frac{1}{2}$, and the probability mass function $b(k)$ is increasing for $k = 0, \ldots, \frac{n}{2} - 1$. (In the extensions, we will examine consequences of distribution asymmetry, $p \neq \frac{1}{2}$.) The difference of posteriors for L-voters and R-voters is evident from $B_L(k; n; p) = B(k - 1; n - 1; p)$ for $k \geqslant 1$ and $B_R(k; n; p) = B(k; n - 1; p)$ for $k \leqslant n - 1$.[9] In particular, see $\frac{b_L(k)}{b(k)} = 2\frac{k}{n}$ and $\frac{b_R(k)}{b(k)} = 2\frac{n-k}{n}$, which implies that posteriors of each type are 'optimistic' in a sense that for type t, events of being in majority (t-majority events) are now seen as being more likely and events of being in minority (t-minority events) are now seen as less likely. For example, L-voters expect any particular L-majority event to be more likely than R-voters, $b_L(k) > b_R(k)$ for $k > n - k$, and vice versa. In brief, each voter considers events where its group is in majority to be relatively more likely, hence her instrumental voting is more affected by gains in the majority-type events.

Each type $t = L, R$ is characterized by the utility function $u_t(c)$ over the elected candidate $c \in K$, where valuation of any elected candidate is invariant to the valuation of another elected candidate. The arguments in the utility function are policy and quality of the elected candidate, and we assume that these arguments are separable. Types are symmetrically antagonistic over the policy, where given separability, $V > 0$ denotes the common relative salience of the policy dimension to the corruption dimension. To introduce a strong tradeoff between policy and candidate's quality, Assumption 1 presumes ideological bias (Krishna and Morgan, 2011); policy dimension is more salient than quality dimension.

Assumption 1 (Policy salience) $V > 1$

[9] For completeness, $B_L(0; n; p) = 0$ and $B_R(n; n; p) = 1$.

Assumption 1 can be interpreted such that voters and the competing parties are sufficiently polarized, hence voters considers the non-valent issue to be of the first-order importance. The salience assumption puts odds against the prospect of high-quality candidates, and accordingly stresses the coordination function of the electoral expectations. By normalizing the benefit from having elected the worst candidate to zero, the voters' objective functions are:

$$u_L(L_1) = V + 1 > u_L(L_0) = V > u_L(R_1) = 1 > u_L(R_0) = 0, \tag{1}$$

$$u_R(R_1) = V + 1 > u_R(R_0) = V > u_R(L_1) = 1 > u_R(L_0) = 0. \tag{2}$$

3.2 Electoral rules and admissible ballots

We consider a subclass of scoring rules that are characterized by the maximal number of positive votes V^+ and the maximal number of negative votes V^-, where votes cannot be cumulated. In any scoring rule, each voter's ballot must be a vector that specifies the number of points that the voter assigns to each candidate. The vectors of points of all voters are summed into a vector of scores, and for an M-seat district, the winning candidates are M candidates with the highest scores. Ties for the M-seat are broken neutrally; a winner of the last seat is chosen randomly among all candidates involved in the tie, each with equal probability. The reason is to make an electoral rule neutral to any other aspect but the ballot structure.[10]

In our setting, each voter of type t submits a ballot $v_t = (v_t^{L_1}, v_t^{L_0}, v_t^{R_1}, v_t^{R_0})$. The scoring rules that we admit have two characteristics: (i) The number of points that the voter gives to a candidate is 1, 0, or -1 (a positive vote, none vote, a negative vote), hence $v_t^c \in \{1, 0, -1\}$. (ii) There are at maximum $V^+ \in \mathbb{N}$ positive votes and at maximum $V^- \in \mathbb{N}$ negative votes on the ballot. (By the *type* of the vote, we mean whether a particular vote is positive or negative.) Hence, the ballot can be truncated. The first characteristic admits $4^3 = 64$ ballots, but the second characteristic reduces the set of feasible ballots. For the simplest rules such as plurality, we have only 5 feasible ballots. For complicated rules such as the 2+1 rule, we have exactly 33 feasible ballots.[11] The largest set of 62 feasible ballots is admitted by combined

[10] In contrast, Meyerson (1993) allows ties to be broken by a secondary voter's ranking. The extra ranking is a technically very useful concept but involves three disadvantages: (i) In reality, the secondary ranking is not available unless list-voting with preferential votes is introduced. (ii) Non-neutral tie-breaking rules will tend to promote candidates with high valence (e.g., high-quality candidates), because in the construction of the secondary ranking, voters will *not risk of any policy loss.* The tradeoff between a policy loss and quality gain is however a crucial tradeoff of the electoral rules. If the aim is to see how the electoral rule itself affects the quality of serious candidates, the incentive to support high-quality candidates must be purely endogenous and the electoral rule should be 'neutral' to the tradeoff. (iii) With the extra assumption in favor of high-quality candidates, it is likely that changes of the electoral rules will effectively have no difference. This will only stem from the fact that the tie-breaking rule will suppress the effects of the electoral rules.

[11] These are permutations of the complete ballot $(1, 1, 0, -1)$ and truncated ballots $(1, 1, 0, 0)$, $(1, 0, 0, 0)$, $(1, 0, 0, -1)$, and $(0, 0, 0, -1)$.

approval-disapproval voting.[12] Additionally, we will restrict ourselves to admissible ballots. A ballot is admissible unless it is weakly dominated by any other feasible ballot.

In our subclass of scoring rules, an electoral rule is a triplet (M, V^+, V^-). We will be examining the following rules: For $M = 1$, consider plurality (1+0 rule), the mixed system (1+1 rule), and approval voting ($V^+ = \#K - 1$; for our quadruplet of candidates, 3+0 rule). For $M = 2$, consider plurality-at-large (2+0 rule), the 2+1 rule, approval voting (3+0), and combined approval-disapproval voting ($V^+ = V^- = \#K - 1$; here 3+3 rule). To sum up, our interest is in how the electoral rule (M, V^+, V^-) affects a fixed electoral situation involving a given set of candidates K, the set of voter's types $\{L, R\}$, and the parameter of the binomial distribution p.

We seek pure-strategy equilibria. A voting profile $\mu(v, t)$ in pure strategies satisfies $\mu(v, t) \in \{0, 1\}$ for any (v, t). The set of feasible pure-strategy voting profiles is a Cartesian product of the sets of feasible ballots and this set is determined by the electoral rule. Throughout the analysis, it will be often useful to alternatively describe the ballot not as a vector of points given to candidates, but as a vector of 'uses' of the feasible votes, hence a vector of $V^+ + V^-$ elements c'. Each vote is either *active* (put on the ballot to some candidate, $c' \in K$) or *inactive* (not put on the ballot, $c' = \varnothing$). We assume that voting is costless, but this does not imply that each voter necessarily actively assigns all feasible votes.[13] Taking inactivity of votes into account, cardinality of the set of the feasible pure strategy profiles is only 25 for plurality rule, but amounts to 999 for the 2+1 rule and $3,844$ for the combined approval-disapproval voting.

Let $S_c(k, \mu)$ be the score of candidate $c \in K$ under voting profile μ. By type-symmetry (see Assumption 2 below), the candidate's score is

$$S_c(k, \mu) = kv_L^c + (n - k)v_R^c.$$

We use the scores to characterize the candidates' probabilities of being elected for given k for a voting profile $\mu(v, t)$. For each candidate and each k, we use an indicator variable $I_c|k = 0, 1$. A candidate's probability of being elected conditional in the event k is $\Pr(I_c = 1|k)$, and his or her *seat probability* is $\sum_{k=0}^{n} \Pr(I_c = 1|k)b(k)$.

For given k, exactly M candidates with the highest scores $S_c(k, \mu)$ win M seats. Let the ordered candidates' scores be $S_1(k, \mu) \geqslant S_2(k, \mu) \geqslant S_3(k, \mu) \geqslant S_4(k, \mu)$. Then, we have $\Pr(I_c = 1|k) = 1$ if $S_c(k, \mu) > S_M(k, \mu)$ and $\Pr(I_c = 1|k) = 0$ if $S_c(k, \mu) < S_M(k, \mu)$. In the case of a tie, recall that all candidates are treated identically, and the seat is allocated randomly. Thus, if $S_c(k, \mu) = S_M(k, \mu)$, then $\Pr(I_c = 1|k) = \frac{1}{z}$ where z is the number of candidates who satisfy $S_c(k, \mu) = S_M(k, \mu)$.

[12] Only ballots $(1, 1, 1, 1)$ and $(-1, -1, -1, -1)$ are not feasible.

[13] Weak dominance does not imply activity of all votes. For a small set of candidates such is our case, a voter can be made strictly worse off by being forced to actively use all votes (c.f., Feddersen and Pesendorfer, 1996).

3.3 Pivotal events and seriousness

For each voting profile μ, we identify the events (i.e., the sets of realizations k) in which an individual voter is decisive, called *pivotal events*. The gains and losses in the pivotal events will shape the voter's best response ballot. More precisely, consider a profile μ and suppose any unilateral deviation of a single voter, characterized by a profile μ'. *Pivotal events* for a pair of profiles (μ, μ') are all events where the vector of candidates' seat probabilities changes. Notice that pivotal events directly depend on the given pair of profiles, and indirectly are rule-specific in a sense that the electoral rule determines which feasible profiles μ and also which alternative profiles μ' are feasible given μ. To avoid excessive notation, we leave the analysis of the pivotal events to each particular electoral rule.

At this stage, it is only valuable to see that for scoring rules where $v_t^c \in \{-1, 0, 1\}$, a pivotal event is either a tie or a near tie for the M-th seat. A *tie* k is characterized by $S_M(k, \mu) = S_{M+1}(k, \mu)$. For any electoral rule we consider, each tie for the M-th seat is obviously a pivotal event. The remaining pivotal events are in *near ties*, where a necessary condition for a near tie is $S_M(k, \mu) - S_{M+1}(k, \mu) \in \{1, 2, 3, 4\}$ or $S_M(k, \mu) - S_{M-1}(k, \mu) \in \{1, 2, 3, 4\}$. The reason to account for differences of at most four points from the score of M-th candidate $S_M(k, \mu)$ is that in the class of rules using non-cumulated $+1$ points and -1 points, a single voter changes an individual candidate's score at maximum by two points (e.g., by adding a positive vote and withdrawing a negative point), and therefore the relative scores of two candidates can not be affected if the difference is by five points or more. Typically, however, the relevant near ties occur only for differences of scores by one or two points.

While district magnitude M does not affect which pairs (μ, μ') are feasible, it affects which events are pivotal. Close races are for the last seat, hence involve M-th and $M+1$-st candidates and possibly also other candidates. Pivotal events depend on $S_M(k, \mu)$, and this function is non-increasing in M. Therefore, although the pivotal events for single-seat districts and two-seat districts can be identical for $M = 1, 2$ in some important profiles, the sets of pivotal events typically differ for single-seat districts and two-seat districts. At this point, notice a difference between single-seat and two-seat districts. For $M = 1$, $S_M(k, \mu)$ is the upper envelope of the score functions, hence is a convex function. As a result, the events where a candidate c wins a seat with a positive probability, $S_c(k, \mu) \geqslant S_1(k, \mu)$, have to constitute a convex set. This property not necessarily holds for $M > 1$.

The set of pivotal events involves all subsets of the sets, including all singletons k. Henceforth, it is convenient to decompose the analysis of gains and loses into analysis of singletons (i.e., by an event, we henceforth mean a singleton k.) For any pivotal singleton k constructed from a pair of profiles (μ, μ'), there must be at least a pair of candidates whose probabilities to get elected at k, $\Pr(I_c = 1|k)$, have changed. Any candidate whose probability of getting elected for a pivotal event k changes is called *a serious candidate in the event k*. A candidate is called *serious* if there exists $k \in N$ such that a candidate is serious in the event k.

Also, we can distinguish between two types of non-serious candidates: a candidate is *strong and not serious* if $S_c(k,\mu) > S_M(k,\mu)$ for any k, and a candidate is *weak and not serious* if $S_c(k,\mu) < S_M(k,\mu)$ for any k.

With the above classification of candidates, we can describe each active vote c' on the ballot (i.e., a positive or negative point) under profile μ either as a *serious* or *non-serious* vote. A serious vote is cast to a serious candidate. By definition, a serious vote affects the seat probability of a corresponding serious candidate. A non-serious vote is a vote cast to a non-serious candidate. Thus, a non-serious vote does not change the seat probability of any candidate.

3.4 Relevant equilibria

Besides focus on pure-strategy equilibria of admissible ballots, Assumption 2 additionally characterizes the equilibrium as symmetric.

Assumption 2 (Symmetry) *Equilibrium ballots are characterized only by types and are type-symmetric.*

The assumption involves two symmetries: *Within-type symmetry* (homogeneity) states that voters of an *identical type* behave in the equilibrium identically; the ballot of any L-type is v_L and the ballot of any R-type is v_R. Unlike Poisson game where payoff-irrelevant type subdivisions have zero effect on marginal probabilities for strategy profiles (Myerson, 1998), multinomial games are not invariant to payoff-irrelevant type subdivisions. Hence, we must directly assume that there is no device that would instruct the voters of the same type to differ in their ballots. *Across-type symmetry* states that L-voter's ballot is type-symmetric to R-voter's ballot, hence $v_R = (v_L^{R_1}, v_L^{R_0}, v_L^{L_1}, v_L^{L_0})$. As a shortcut, we will henceforth represent each equilibrium voting profile only by the ballot v_L.

We describe the approach to identification of the equilibria:

1. We construct the set of symmetric pure-strategy voting profiles. We eliminate profiles with apparently inadmissible ballots[14] involving a positive vote to the worst candidate ($v_L^{R_0} = v_R^{L_0} = 1$) or a negative vote to the best candidate ($v_L^{L_1} = v_R^{R_1} = -1$) in order to obtain the set of candidate strategy profiles $\mathcal{V}$.

2. For each profile $\mu \in \mathcal{V}$, we derive the corresponding candidates' score functions $S_c(k,\mu), c \in K$, and the function $S_M(k,\mu)$.

3. For the score functions, we identify pivotal events that are either ties or near ties for the M-th seat. This is relatively straightforward given that the score functions are linear in k.

[14] An alternative ballot that would make the vote inactive would weakly dominate this ballot. For other ballots, admissibility can be comprehensively evaluated only by constructing score functions for all feasible profiles (v_L, v_R), and identifying all pivotal events from all realizations $k \in N$.

4. The pivotal events identify the sets of serious candidates and non-serious candidates.

5. We check whether the voting strategies are best responses. That is, for each type t, we consider all admissible unilateral deviations. Relevant deviations are such that there exists a pair (c, k) where $\Pr(I_c = 1|k)$ changes.

6. For each relevant deviation, we calculate each voter's the expected gain (positive or negative) at each pivotal singleton k where the candidates' probabilities of being elected have changed. The total expected gain is then the weighted sum of the expected gains times posterior probabilities of the pivotal singletons, $b_t(k)$. A profile μ is an equilibrium only if each total expected gain is non-positive.

The relevant equilibria are weak or strict. We call any weak equilibrium where the equilibrium best response is the entire set of all feasible ballots a *passive equilibrium*. In a passive equilibrium, none candidate is serious. Although we admit such equilibria for the sake of completeness, we must bear in mind that passive equilibria would disappear with any infinitesimal cost of voting. The main reason to admit passive equilibria is that in our electoral situation in two-member districts, passive equilibria involve an interesting case of successful coordination of both groups along the quality dimension.

4 Single-member districts

For both single-seat and two-seat districts ($M = 1, 2$), we will always consider a triplet of electoral rules. The simplest rule assigns exactly $V^+ = M$ positive votes. The modified rule adds a single negative vote, $V^- = 1$. Third, we consider approval voting with $V^+ = 3$ positive votes.

The size of the set K deserves a special note for single-member districts. If one thinks that each group will be represented by a single party, and each party nominates M candidates for M seats, then we have only one nominee for each group in single-member districts. This scenario is possible. Therein, valence competition shifts from elections into party primaries. Trivially, elections are pairwise voting under strict preferences where all scoring electoral rules are strategically equivalent.

When characterizing electoral outcomes generated in relevant equilibria of the rules, we will be primarily checking the events when an electoral rule assigns in the equilibrium a seat to Condorcet winner and the events when Condorcet loser gains a seat. With two types of voters, Condorcet winner and Condorcet loser are defined by the preferences of the dominant group of voters. For $k \leqslant \frac{n}{2}$, R_1 is Condorcet winner and L_0 is Condorcet loser. For $k \geqslant \frac{n}{2}$, L_1 is Condorcet winner and R_0 is Condorcet loser.

4.1 Plurality

Plurality rule (1+0) exhibits a classic coordination problem with multiple relevant equilibria: Either low-quality candidates $\{L_0, R_0\}$ compete against each other, or high-quality candidates $\{L_1, R_1\}$ compete against each other. Both equilibria share a unique pivotal tie $k = \frac{n}{2}$ with a pair of serious candidates, hence there is a close race only if the populations of R-voters and L-voters are balanced. The other candidates are non-serious. The equilibrium best response is always to actively support the better of the two serious candidates. (For proofs, see always Appendix.)

Proposition 1 (Plurality) *For plurality, there are two strict equilibria: (i) L-voters support L_1 and R-voters support R_1. (ii) L-voters support L_0 and R-voters support R_0.*

Both equilibria are strict, hence the single positive vote is always active and serious. The first equilibrium is associated with a sincere ballot $v_L = (1, 0, 0, 0)$. For any k, Condorcet winner is elected and Condorcet loser is not elected. The second equilibrium is associated with a non-sincere ballot $v_L = (0, 1, 0, 0)$, but voting is sincere over the subset of serious candidates $K' = \{L_0, R_0\}$. For any k, Condorcet winner is a non-serious candidate, hence is not elected. Condorcet loser is surely elected in tie $k = \frac{n}{2}$. The first equilibrium Pareto-dominates the second equilibrium.

4.2 Mixed system

What is the effect of adding negative votes on the ballot? The numbers and types of the available votes affect the structure of the close race primarily (but not exclusively) through the effect on the number of candidates that are understood to be serious or viable. Myerson (1999) develops a quick tatonnement argument to illustrate the difference between positive and negative votes: For positive votes, serious candidates attract positive points, hence high seriousness coincides with high strength and vice versa. Low-ranked candidates are not considered serious and correspondingly remain weak. In contrast, for negative votes, serious candidates attract negative points, hence high seriousness coincides with low strength. There is an equilibrium only if all candidates are similarly serious and strong, and the negative points are dispersed over a large pool of candidates.

In the mixed system (1+1), we observe that the effect of a negative vote is indeed to generate a large pool of serious candidates. More specifically, the voters tend to cast positive and negative votes to a pair of serious candidates. Negative points from one group then cancel out positive points from the opposing group. As a result, at the tie $k = \frac{n}{2}$, serious candidates have scores equal zero. But the other two candidates have also scores equal zero, hence must also be considered serious.

With all candidates considered serious, strategic mixing becomes very likely. Mixing then implies that a low-quality candidate wins seats with a positive probability. Proposition 2 proves that the incentive to mix is absent in a configuration with all candidates being serious only in a special case when the pair (L_1, R_1) attracts both positive and negative votes and ideological bias (salience of policy dimension) is very low. In such circumstances, consider L-voter's best deviation. L-voter contemplates transferring a positive vote from L_1 to L_0. Thereby, L-voter wins tie $k = \frac{n}{2}$ (i.e., a seat for L_0 rather than the lottery over the four candidates) but induces a lottery over L_1 and L_0 at near tie $k = \frac{n}{2} + 1$ (i.e., a decrease in the quality). This deviation does not pay off only if the policy gain at the tie is too small to compensate for the quality loss at the near tie. This requires a sufficiently small ideological bias V.

Proposition 2 (Mixed system) *In the mixed system, a relevant equilibrium exists if and only if $V \leqslant 2$. In the relevant equilibrium, L-voters cast positive votes to L_1 and negative votes to R_1, and R-voters cast positive votes to R_1 and negative votes to L_1.*

The L-voter's strategy characterizing the unique relevant pure-strategy profile is $v_L = (1, 0, -1, 0)$. The set of elected candidates in the relevant equilibrium is identical to the set for the Pareto-superior equilibrium from plurality. We may conclude that in comparison with plurality, the presence of the negative vote has helped to eliminate the Pareto-inferior equilibrium, but only in a small parametrical subspace. In the remaining cases, where both groups focus on reducing chances of the opposing candidates, they cancel out each other's votes, and there is a window of opportunity for the weak and non-serious candidates with strategic mixing as a result. The resulting mechanics of mixing is very close to the Myerson's (1993) analysis of the effect of Borda Count on the quality of elected candidates.

4.3 Approval voting for single-member districts

Myerson (1993) argues that approval voting is fully effective in a sense that for every electoral situation that involves high-quality and low-quality candidates of the same policy type, all equilibria give low-quality candidates zero seat probability. Effectiveness of the approval voting however hangs on the selected tie-breaking rule which does not treat identically those candidates who have obtained the same number of votes. Myerson (1993) highlights that once ties are resolved randomly, the favorable result no longer holds in general.

In our particular electoral situation, admissibility is sufficient to yield a unique favorable equilibrium under approval voting. The reason is that the strategy $v_L = (0, 1, 0, 0)$ known from the Pareto-inferior equilibrium is weakly-dominated by $v_L = (1, 1, 0, 0)$, hence becomes inadmissible.

Proposition 3 (Approval voting, $M = 1$) *For approval voting in single-member districts, L-voters support L_1 and R-voters support R_1.*

The equilibrium ballot under approval voting is identical to the Pareto-superior ballot in plurality, but in contrast to plurality, the equilibrium now involves two inactive votes. Thus, the equilibrium is not strict, but weak. Weakness may be considered as a disadvantage if we consider an equilibrium selection criterion that dictates that votes are active unless inactivity brings a positive gain.

5 Two-member districts

In this section, we proceed to the main analysis of double-member districts. We will discuss both quality and representativeness of the candidates. So far, the only performance criterion for electoral rules were the seat probabilities of the low-quality candidates. For two-seat districts, another natural criterion is how an electoral system protects minority interests. Combined together, we simply ask for which rule and under what conditions the second seat is allocated to a high-quality candidate of the minority group.

5.1 Plurality-at-large

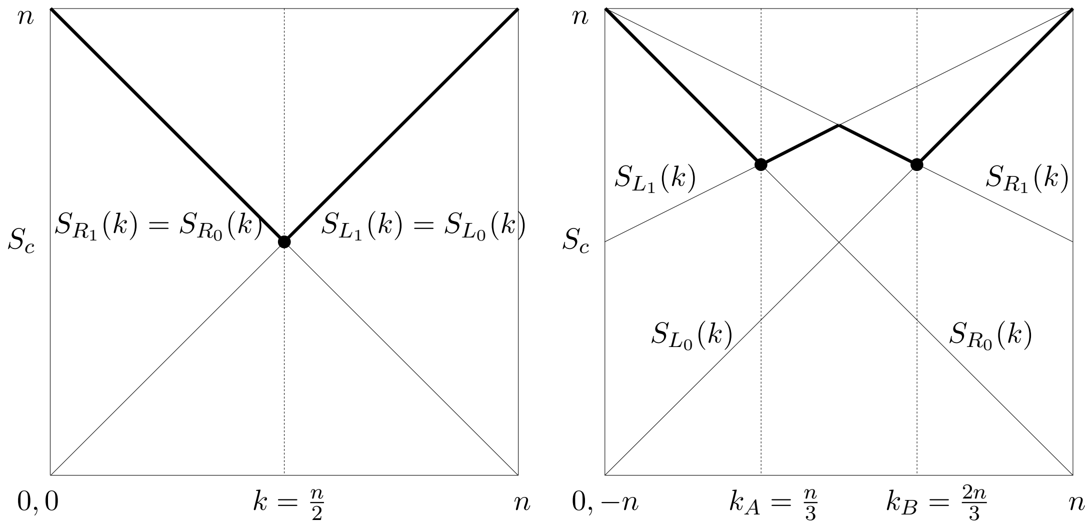
If two positive votes can be cast (plurality-at-large, dual voting, block voting), Proposition 4 derives two equilibria, both characterized by all votes being active. The first equilibrium is characterized by $v_L = (1, 1, 0, 0)$, and the second equilibrium by $v_L = (1, 0, 1, 0)$. The two equilibria are structurally entirely different. The first equilibrium is sincere and strict, hence all votes are active and serious, and voting is sincere. The second equilibrium is non-sincere and passive, hence all votes are non-serious.

Proposition 4 (Plurality-at-large) *For plurality-at-large, we have two relevant equilibria: (i) In a strict equilibrium, L-voters support* (L_1, L_0) *and R-voters support* (R_1, R_0)*. (ii) In a passive equilibrium, all voters support* (L_1, R_1)*.*

When discussing the 2+0 rule, we will focus on the strict equilibrium because of its strictness and sincereness. The strict equilibrium is characterized such that in (almost) every event, only one party wins both seats. Hence, minorities are not represented. Quality is not high either; the second seat is always for a candidate of the inferior quality.

5.2 The 2+1 electoral rule

First, the passive equilibrium of the 2+0 rule remains an equilibrium even if a single negative vote is available. The reason is that with an extension of the set of admissible ballots by negative votes, all candidates still remain non-serious, and there is no strict incentive to cast a negative vote. The only difference is that the ballot $v_L = (1, 0, 1, 0)$ becomes weakly dominated by the ballot $v_L = (1, 0, 1, -1)$, hence the latter ballot now characterizes the unique passive equilibrium in admissible strategies. In addition to a passive equilibrium, Proposition 5 identifies a unique sincere and strict equilibrium where all candidates are serious.



(a) The 2+0 rule (b) The 2+1 rule

Figure 1: Candidates' scores in the strict equilibria

Proposition 5 (2+1) *For the 2+1 rule, we have at most two relevant equilibria: (i) Iff $V \leqslant 3$, there exists a strict equilibrium where L-voters support (L_1, L_0) and punish R_0, and R-voters support (R_1, R_0) and punish L_0. (ii) In a passive equilibrium, all voters support (L_1, R_1); L-voters punish R_0 and R-voters punish L_0.*

Figure 1 illustrates the electoral outcomes achieved in the strict equilibrium of the 2+1 rule and compares them with the outcomes of the strict equilibrium of the 2+0 rule.

In the strict equilibrium of the 2+1 rule, voters focus on punishing the weaker candidate from the other camp instead of punishing the stronger candidate. Why? There are two effects. The first effect concerns *nominal win gains in the pivotal events*. In the equilibrium, two pivotal ties $(k_A, k_B) = (\frac{n}{3}, \frac{2n}{3})$ arise. Consider L-voters: (i) If L-group is in majority ($k = k_B$), her weaker candidate L_0 competes with the stronger candidate R_1 for the second seat. The nominal gain of getting L_0 instead of R_1 is for L-voter $V - 1$. (ii) If L-group is in minority ($k = k_A$), her stronger candidate L_1 competes with the weaker candidate R_0 for the second seat. The nominal gain of getting L_1 instead of R_0 is for L-voter $V + 1$. *The nominal gain associated with the minority event is more valuable*, since $V + 1 > V - 1$. The second effect concerns relative frequencies of the pivotal events. We know that $\frac{b_L(k)}{b_L(n-k)} = \frac{n-k}{k}$, hence minority event k_A is seen as more likely than majority event k_B, $b_L(k_B) = 2b_L(k_A)$.

To compare the strict equilibria of the 2+1 rule and the 2+0 rule:

- Both strict equilibria can be intuitively interpreted such that the positive votes are cast along the more salient dimension, and the negative vote is cast along the less salient dimension.

- The 2+1 rule increases the expected quality of the elected candidates for the intermediate levels $\frac{n}{3} \leqslant k \leqslant \frac{2n}{3}$. Thus, the rule does not abolish electoral prospects of the low-quality candidates altogether, but a low-quality candidate never wins the first seat and competes only in races over the second seat. In fact, the presence of low-quality candidates among serious candidates has a desirable function of attracting negative votes that would be otherwise targeted to high-quality candidates.

- The 2+1 rule assigns the second seat to the high-quality candidate of the minority group if the ex post differences in sizes of the groups are not too large, namely for $\frac{n}{3} < k < \frac{2n}{3}$.

- The 2+1 rule not only guarantees that Condorcet winner wins the seat, but also never elects a Condorcet loser. In the 2+0 rule, Condorcet losers L_0 and R_0 were serious candidates for a unique tie $k = \frac{n}{2}$, and gained the seat with a positive probability. In the 2+1 rule, L_0 (respectively R_0) is serious only in the events where he or she is *not* Condorcet loser.

- The existence of multiple ties is sustained because of the unequal number of non-cumulated positive and negative votes ($V^+ > V^-$). The unequal number generates a difference between positive and negative votes in *substitutability/transferrability of votes across events*. While a negative vote can be transferred between candidates from the opposing group, a positive vote cannot be transferred because positive votes cannot be cumulated. If positive votes could be freely transferred, that the voter would deviate by shifting all positive and negative votes to the more important pivotal events, but that would be inconsistent with its existence. The lack of transferability of the positive votes associated with impossibility to cumulate votes is thus one of the necessary conditions for this equilibrium.

5.3 Approval voting

The beneficial effect of the 2+1 rule can be attributed to the extended scope for the voter's discrimination associated with additional negative vote. Once voters protected their primary policy interests by means of two votes, they could secure their secondary quality interests by an extra vote. Yet, a crucial difference is whether the extra vote is positive or negative. If the extra third vote is positive such as in approval voting, the restriction that points of a single type cannot be cumulated to a single candidate binds. In contrast, this restriction of cumulation does not bind the use of an extra negative vote.

By Proposition 6, approval voting normally replicates equilibria from the 2+0 rule. Only in the special case of extremely low policy salience, an additional equilibrium emerges, where all three positive votes become active, and the first three candidates are supported at the same time. Nevertheless, this extra equilibrium vanishes in a large electorate as $\lim_{n\to\infty} \hat{V} = 1$.

To sum up, in a large electorate in our electoral situation, approval voting tends to generate identical equilibria like plurality-at-large.

Proposition 6 (Approval voting, $M = 2$) *For approval voting in two-member districts, there are at most three relevant equilibria: (i) In a strict equilibrium, L-voters support (L_1, L_0) and R-voters support (R_1, R_0). (ii) In a passive equilibrium, all voters support (L_1, R_1). (iii) If and only if $V \leqslant \hat{V}$, where $\hat{V} := \frac{3n+2}{3n+1} \geqslant 1$, there exists a strict equilibrium where L-voters support (L_1, L_0, R_1) and R-voters support (R_1, R_0, L_1).*

5.4 Combined approval-disapproval voting

The electoral rule with the largest set of feasible profiles $\mathcal{V}$ is combined approval-disapproval voting, here 3+3 rule. This rule allows for an arbitrary combination of positive and negative votes. In our electoral situation, the 3+3 rule generates a single strict and sincere equilibrium and three passive equilibria. In the strict equilibrium, electoral competition reduces only to the policy dimension, exactly as in the 2+0 rule. All passive equilibria yield an identical and favorable electoral outcome featuring coordination upon high-quality candidates, but recall that in a passive equilibrium, none vote is serious.

Proposition 7 (Combined approval-disapproval voting, $M = 2$) *For combined approval-disapproval voting in two-member districts, there are four relevant equilibria: (i) In a strict equilibrium, L-voters support (L_1, L_0) and punish (R_1, R_0). R-voters support (R_1, R_0) and punish (L_1, L_0). In the passive equilibria: (ii) All voters support (L_1, R_1) and punish (L_0, R_0). (iii) All voters support (L_1, R_1). L-voters punish R_0 and R-voters punish L_0. (iv) L-voters support L_1 and R-voters support R_1. All voters punish (L_0, R_0).*

5.5 Comparison of the electoral rules

We have identified a set of relevant equilibrium voting profiles that occur under selected scoring electoral rules. We now compare the electoral rules in three steps. First, for each of the voting profiles μ, we assess quality and representativeness of electoral outcomes. Second, we link these profiles to the rules and examine which particular profile is most realistic for each of the rules. For that purpose, we assess all equilibrium profiles by a set of selection criteria. Third, we will compare performance of the rules, with a primary focus on rules for $M = 2$.

Table 2 lists several quality and representativeness properties of the elected M candidates. More specifically, we write down the set of realizations k for which a given property holds under a respective voting profile μ. Begin with a property *All-CW* (all Condorcet winners). In *All-CW* events, all elected candidates are Condorcet winners with certainty. In *CW* events, Condorcet winner is at least one of the candidates with a positive probability. *CL* event is when Condorcet loser is at least one of the candidates with a positive probability. *All-Quality* event is when all elected candidates are high-quality candidates with certainty, whereas *Quality* event is when a high-quality candidate is at least one of the elected candidates with a positive probability. *Minority* event is when a candidate from the minority group is at least one of the elected candidates with a positive probability.[15] Notice that in the intersection of *Minority* and *Quality* events, the second seat is allocated to a high-quality candidate of the minority group.

From the welfare point of view, a profile yields unambiguously better results if a set of realizations k for which Condorcet winner properties and Quality properties hold is larger. For Condorcet loser property, the smaller is the set of the complying events, the better is the profile. For Minority property, we will explicitly compute utilitarian welfare in Section 5.6.

Table 2: Properties of the elected candidates in the relevant profiles

Seats	v_L	All-CW	CW	CL	All-Quality	Quality	Minority
$M=1$	$(1,0,0,0)$	N	N	$\varnothing$	N	N	$\varnothing$
	$(0,1,0,0)$	$\varnothing$	$\varnothing$	$\{\frac{n}{2}\}$	$\varnothing$	$\varnothing$	$\varnothing$
	$(1,0,-1,0)$	$N\setminus\{\frac{n}{2}\}$	N	$\{\frac{n}{2}\}$	$N\setminus\{\frac{n}{2}\}$	N	$\varnothing$
$M=2$	$(1,1,0,0)$	$\varnothing$	N	$\{\frac{n}{2}\}$	$\varnothing$	N	$\varnothing$
	$(1,1,-1,-1)$	$\varnothing$	N	$\{\frac{n}{2}\}$	$\varnothing$	N	$\varnothing$
	$(1,0,1,0)$	$\{\frac{n}{2}\}$	N	$\varnothing$	N	N	N
	$(1,0,1,-1)$	$\{\frac{n}{2}\}$	N	$\varnothing$	N	N	N
	$(1,-1,1,-1)$	$\{\frac{n}{2}\}$	N	$\varnothing$	N	N	N
	$(1,-1,0,-1)$	$\{\frac{n}{2}\}$	N	$\varnothing$	N	N	N
	$(1,1,0,-1)$	$\{\frac{n}{2}\}$	N	$\varnothing$	$\{\frac{n}{3}+1,\dots,\frac{2n}{3}-1\}$	N	$\{\frac{n}{3},\dots,\frac{2n}{3}\}$
	$(1,1,1,0)$	$\{\frac{n}{2}\}$	N	$\varnothing$	$\{1,\dots,n-1\}$	N	$\varnothing$

The second step is to evaluate properties of the voting profiles under given electoral rules. The electoral rule affects properties of the voting profile because with a profile-preserving electoral rule change, the number of inactive votes changes. And it is the number of inactive votes that determines, amongst others, whether the equilibrium is strict or weak. We define the following set of selection criteria:

- *Activeness*: All votes are active or all candidates receive votes. Put formally, the number of active votes is $\min\{V^+ + V^-, \#K\}$. (In other words, either $c' \neq \varnothing$ for any available vote or $c' = \varnothing$ and $v_t^c \neq 0, \forall c \in K$.) Notice that a strict equilibrium does not have to have all votes active.

- *Seriousness*: All active votes are serious. The idea is that a voter does not mix instrumental and non-instrumental votes on the ballot. This condition is obviously always met in a strict equilibrium and is never met in a passive equilibrium.

[15] When groups are of equal sizes, none of the groups is considered to be in minority.

- *Strictness*: All active votes are cast strictly, hence changing any active vote differently (either by switching to a different candidate or making it inactive) strictly decreases the payoff. A strict equilibrium implies Strictness, but not vice versa. Similarly, Strictness is sufficient but not necessary for Seriousness.

- *Honesty*: The voter casts positive votes to v^+ best candidates and negative votes to v^- worst candidates, where (i) $(v^+, v^-) = (V^+, V^-)$ if $V^+ + V^- \leqslant \#K$, and (ii) $v^+ + v^- = \#K$ if $V^+ + V^- > \#K$. This version of sincereness is motivated by two aspects: (i) First, voters are sincere in a sense that if a candidate $a \in K$ receives from t-voter more points than a candidate $b \in K \setminus \{a\}$, $v_t^a > v_t^b$, then the t-voter strictly prefers candidate B to A, $u_t(a) > u_t(b)$. (Recall that by $V > 1$, voters have only strict preferences.) (ii) Second, out of all ballots that satisfy the first aspect, the voter selects the one which uses the maximal possible amount of votes. For example, honest approval voting is in fact disapproval voting by this criterion. Additionally, combined approval-disapproval voting is honest only if all candidates receive votes; consequently, in our definition, honest voters separate candidates into two groups, not three groups.

- *M-honesty*: In contrast to Honesty, each voter considers only votes to serious candidates, and takes the number of seats M into account. This reflects the following heuristic: (i) Expect a set of serious candidates $K' \subseteq K$. (ii) Vote honestly in the set of serious candidates in order to support exactly M most preferred serious candidates. The remaining votes are inactive. Thus, if the set of the serious candidates is sufficiently small or if M is small, an M-honest profile requires some votes to remain inactive.

Table 3 evaluates the voting profiles by the selection criteria.

Table 3: Compliance of the relevant equilibria with the selection criteria

Seats	Rule	Profile (v_L)	Equilibrium	Activeness	Seriousness	Strictness	Honesty	M-honesty
$M = 1$	1+0	$(1, 0, 0, 0)$	strict	x	x	x	x	x
		$(0, 1, 0, 0)$	strict	x	x	x		x
	1+1	$(1, 0, -1, 0)$	strict	x	x	x		
	3+0	$(1, 0, 0, 0)$	weak		x	x		x
$M = 2$	2+0	$(1, 1, 0, 0)$	strict	x	x	x	x	x
		$(1, 0, 1, 0)$	passive	x				
	2+1	$(1, 1, 0, -1)$	strict	x	x	x	x	x
		$(1, 0, 1, -1)$	passive	x				
	3+0	$(1, 1, 0, 0)$	strict		x	x		x
		$(1, 0, 1, 0)$	passive					
		$(1, 1, 1, 0)$	strict	x	x	x	x	
	3+3	$(1, 0, 1, -1)$	passive					
		$(1, -1, 0, -1)$	passive					
		$(1, -1, 1, -1)$	passive	x				

Table 3: Compliance of the relevant equilibria with the selection criteria

$(1,1,-1,-1)$	strict	x	x	x	x	x

Only four profiles meet all five criteria: $(1,0,0,0)$ for 1+0, $(1,1,0,0)$ for 2+0, $(1,1,0,-1)$ for 2+1, and $(1,1,-1,-1)$ for 3+3. If we replace *Honesty* by *M-honesty*, we add $(0,1,0,0)$ for 1+0. Considering only these five relevant profiles, we can now proceed to interpreting how the structure of the electoral rules affects the voter's tradeoffs between quality and ideology:

- Plurality: By Duverger Law, the set of serious candidates is restricted. Strategic voting is motivated by the voter's primary focus upon the serious candidates. This leads to self-fulfilling prophecies and a well-known coordination failure that may protect low-quality candidates. The coordination problem can be addressed only by restricting the number of candidates of a group by party nominations, hence to shifting valence competition from the election stage to a pre-election stage.
- Plurality-at-large: Voters are relieved of the primary coordination problem and are motivated to elect both candidates from their camp. But there is no discrimination along the quality dimension, and the legislature is of mixed quality. There is no incentive for non-sincere voting because of a single tie and all candidates being serious candidates in that tie.
- The 2+1 rule: Voters primarily support candidates that promote their favorable platform, and secondarily have an incentive to oust low-quality candidate of the other group. The key novelty is the existence of *multiple ties* where two different pairs of candidates compete with each other. Each tie (and the corresponding pivotal events) is of a different importance for each group of the voters; the higher relative value of the t-minority event motivates each t-voter to cast the negative vote to the low-quality candidate of the other group.
- Approval voting: Approval voting replicates the Pareto-superior outcome for plurality (under $M = 1$) and the strict outcome for plurality-at-large (under $M = 2$). Thus, it has favorable properties if compared to conventional electoral rules in single-member districts. On the other hand, compared to the 2+1 rule, the voters cannot discriminate in two degrees in double-member districts.
- Combined approval-disapproval voting: The voters are given the option to discriminate in two dimensions by arbitrarily mixing positive and negative votes but they strategically 'overstate' their preferences and discriminate only by a single degree. Under $M = 2$, the strict equilibrium only replicates plurality-at-large.

5.6 Welfare for the two-seat electoral rules

In two-seat districts, two electoral outcomes emerge out of the strict and realistic voting profiles: In *non-mixed electoral outcome* (denoted $O = O_n$), the majority group always elects their first two candidates (e.g., a strict equilibrium in the 2+0 rule). In *mixed electoral outcome* (denoted $O = O_m$), candidates from both groups are elected only if the differences in group sizes are not too large (i.e., a strict equilibrium in the 2+1 rule). How do these outcomes rank in terms of utilitarian welfare?

The two outcomes identically allocate the first seat to the Condorcet winner (R_1 if $k < \frac{n}{2}$ and L_1 if $k > \frac{n}{2}$; for $k = \frac{n}{2}$, there is a tie). The effective comparison is only about the second seat. While both outcomes elect R_0 for the second seat for $k < \frac{n}{3}$ and L_0 for $k > \frac{2n}{3}$, the mixed outcome elects L_1 (or R_1) for $\frac{n}{3} < k < \frac{2n}{3}$ instead of R_0 (or L_0).

For convenience, we compare welfare of the two outcomes in the limit of $n \to \infty$. Specifically, let $\phi := \frac{k}{n}$. For $n \to \infty$, $F(\phi)$ that is the asymptotical distribution of the binomial distribution $B(k)$ for the size of voters normalized to unity, which is a normal distribution on the domain $\phi \in [0, 1]$, with mean $\frac{1}{2}$ and standard deviation $\frac{1}{4}$.

For any outcome $O \in \{O_n, O_m\}$, the expected utilitarian welfare from the second seat is

$$W(O) \equiv \int_0^1 \left[\phi u_L(c_2(O, \phi)) + (1 - \phi) u_R(c_2(O, \phi))\right] f(\phi) d\phi, \tag{3}$$

where $c_2(O, \phi)$ denotes the second elected candidate for given ϕ under given outcome O.

To start with, we derive the socially optimal candidate for each $\phi \leqslant \frac{1}{2}$. Given the absence of R_1 candidate, we call the socially optimal candidate for the second seat as the *second-best candidate*. The second-best candidate is

$$c_2^*(\phi) = \arg \max_{\{R_0, L_1, L_0\}} \phi u_L(c) + (1 - \phi) u_R(c). \tag{4}$$

Trivially, since $u_t(L_1) > u_t(L_0)$ for both types, the second-best candidate must be either R_0 or L_1 for $\phi \leqslant \frac{1}{2}$. The key inequality characterizing the second-best candidate is

$$\phi u_L(L_1) + (1 - \phi) u_R(L_1) = 1 + \phi V \lesseqgtr (1 - \phi) V = \phi u_L(R_0) + (1 - \phi) u_R(R_0). \tag{5}$$

There is a threshold level $\hat{\phi} := \frac{V-1}{2V}$ which defines the second-best efficient candidate:

$$c_2^*(\phi) = \begin{cases} R_0 & \text{if } \phi \leqslant \hat{\phi} \\ L_1 & \text{if } \hat{\phi} \leqslant \phi \leqslant \frac{1}{2} \\ R_1 & \text{if } \frac{1}{2} \leqslant \phi \leqslant 1 - \hat{\phi} \\ L_0 & \text{if } 1 - \hat{\phi} \leqslant \phi \end{cases} \tag{6}$$

Later, it will be useful to see that the threshold satisfies

$$\hat{\phi}(V + 1) = (1 - \hat{\phi})(V - 1). \tag{7}$$

The threshold has an intuitive comparative statics: With an increasing relative importance of quality (decreasing V), $\hat{\phi}$ decreases. It is socially more important to establish a high-quality (but minority) candidate than a majority (but low-quality) candidate.

Proposition 8 proves that under a necessary condition for the existence of a mixed outcome (i.e., the existence of a strict equilibrium under the 2+1 rule), the mixed outcome more frequently elects the second-best efficient candidate, hence welfare-dominates the non-mixed outcome. As a corollary, if a strict equilibrium exists for the 2+1 rule, then the 2+1 rule maximizes welfare relative to the other rules in two-member districts.

Proposition 8 (Welfare) *If a strict equilibrium exists for the 2+1 rule, then the mixed electoral outcome welfare-dominates the non-mixed electoral outcome,* $W(O_m) \geqslant W(O_n)$.

6 Extensions

6.1 Manipulating the strength of the negative vote

In our class of our scoring rules, each positive vote adds +1 to the candidate's score, and each negative vote adds −1 to the score. We may augment the 2+1 rule by considering that each negative point adds $-m$ to the score, where $m \geqslant 0$. For $m = 0$, the augmented 2+1 rule is equivalent to the 2+0 rule. We will investigate the effect of increasing m on the electoral outcomes generated by sincere profile, $v_L = (1, 1, 0, -m)$. Under sincere profile, the pair of pivotal events (k_A, k_B), where $k_A + k_B = n$, have the following structure:[16]

$$(k_A, k_B) = \left(\frac{n}{m+2}, \frac{(m+1)n}{m+2} \right). \tag{8}$$

If sincere profile is an equilibrium, then increasing m increases the interval of mixed outcomes, $k \in [k_A, k_B]$. Minority group is more represented, but as a side effect, there is also a larger difference in posteriors between the types and the larger differences constrain the existence of the sincere equilibrium. Put vice versa, by decreasing m, one relaxes the existence constraint and thereby secures the existence of a sincere equilibrium even for very large V, but only at the cost of reduced representativeness of outcomes. In this section, we investigate welfare effects of manipulation of the existence constraint and representativeness of outcomes.

Let $\bar{m}(V)$ be *the existence constraint*, namely the highest m that secures the existence of a sincere equilibrium. That is, $m \leqslant \bar{m}(V) : b_L(k_A)(V+1) \geqslant b_L(k_B)(V-1)$. Using $\frac{b_L(k_A)}{b_L(k_B)} = \frac{k_A}{n-k_A}$, a sincere equilibrium exists if

$$k_A(V+1) \geqslant (n - k_A)(V-1). \tag{9}$$

[16] Here, we assume that (m, n) are set such that both $\frac{n}{m+2}$ and $\frac{(m+1)n}{m+2}$ are integers.

Entering $\frac{n-k_A}{k_A} = m + 1$ from (8) into the existence condition in (9), we obtain $m \leqslant \frac{2}{V-1}$, hence the existence constraint is $\bar{m}(V) = \frac{2}{V-1}$. Clearly, with a larger ideological bias, a sincere equilibrium is incompatible with largely discriminative negative vote.

How does a change in m affect welfare? From Section 5.6, a welfare-maximizing profile is characterized by mixing in the interval $\hat{\phi} \leqslant \frac{k}{n} \leqslant 1 - \hat{\phi}$. Let $m^*(V)$ be the *welfare-maximizing value*, $m^*(V) : k_A = \hat{\phi} n$. From (8), we know that k_A is decreasing in m. Thus, welfare will be increasing in $m < m^*(V)$ and decreasing in $m > m^*(V)$.

At the welfare-maximizing value $m^*(V)$, we obtain $\frac{k_A}{n-k_A} = \frac{\hat{\phi}}{1-\hat{\phi}}$, or alternatively

$$k_A(V + 1) = (n - k_A)(V - 1). \tag{10}$$

Notice that (10) is a discrete version of (7). Clearly, the condition holds if and only if (9) is satisfied with equality. Consequently,

$$\bar{m}(V) = m^*(V). \tag{11}$$

Interestingly, the welfare-maximizing value $m^*(V)$ always secures that the sincere profile is an equilibrium, and in fact *the welfare-maximizing value of negative discrimination is the largest level of discrimination that is compatible with a sincere equilibrium.* The threshold level of k_A where an ideological mix of candidates is socially optimal is at $k_A[u_L(L_1) - u_L(R_0)] = (n - k_A)[u_R(R_0) - u_R(L_1)]$, or $k_A(V + 1) = (n - k_A)(V - 1)$, and exactly the same condition characterizes L-voter's indifference over the use of the negative vote.

6.2 Asymmetric districts for the 2+1 rule

Consider an ex ante leftist district, $p > \frac{1}{2}$. L-group is an *ex ante majority* and R-group is an *ex ante minority.* In an asymmetric district, we must consider type-asymmetric equilibria. That is, we relax Assumption 2 and consider only within-type symmetry. For the sincere profile of the 2+1 rule, the event $k_A = \frac{n}{3}$ is now very unlikely, whereas the event $k_B = \frac{2n}{3}$ is extremely likely, and the relative frequencies increase in p, as $b(\frac{n}{3})$ is decreasing in p and $b(\frac{2n}{3})$ is up to a point increasing in p.

The ratio of the frequencies of the two events matters significantly, as it influences each t-voter's decision which of the two events to address with a single negative vote. For ex-ante-minority voters, sincere voting remains the best response because the more valuable minority event is also an increasingly more likely event. For ex-ante-majority voters, however, the more valuable minority event (winning the first seat) becomes very unlikely, and the voters consider using the negative vote rather in the more likely (albeit less valuable) majority event.

As a result, ex-ante-majority L-voters consider targeting the stronger candidate of the ex-ante-minority R-group. At the same time, the strength and seriousness of both R-candidates is upon votes of L-group. Thus, when L-voters reduce chances of the stronger R-candidate, they in fact improve chances of the weaker R-candidate, and the gap between the stronger and weaker candidates narrows. In equilibrium, L-voters mix the negative vote, which is the classic underdog property of negative voting. Underdog effect is present whenever the voters have an incentive to spread support across many candidates or parties, and this effect protects the vote share of the weaker candidates.

Proposition 9 proves that in the equilibrium, ex-ante-minority voters are sincere, and ex-ante-majority voters mix, with a negative vote *more likely* to the low-quality candidate R_0.

Proposition 9 (2+1, asymmetry) *For $p > \frac{1}{2}$ such that $b_L(\frac{n}{3})(V+1) < b_L(\frac{2n}{3})(V-1)$, the strict equilibrium in the 2+1 rule involves a sincere ballot $v_R = (1,1,0,-1)$ and mixing of a sincere ballot $v_L = (1,1,0,-1)$ with probability $\alpha \in (\frac{1}{2},1)$ and a strategic ballot $v_L = (1,1,-1,0)$ with probability $1-\alpha \in (0,\frac{1}{2})$.*

What are properties of the mixed equilibrium? Importantly, on the average, the quality is still better than in the strict equilibrium at 2+0 rule, and minorities are also better protected. Thus, the underdog effect only limits the beneficial effect of the negative vote in terms of quality, but the effect still remains positive. By any criterion, the outcome of the mixing profile for the 2+1 rule lies *in-between* the non-mixed outcome O_n and the mixed outcome O_m.

In Section 2.4, we have derived that a finite Poisson game is equivalent to our finite binomial setting if $p = \frac{1}{2}$. We now derive an approximation of the mixed strategy equilibrium for $p \neq \frac{1}{2}$ in a finite Poisson game and demonstrate that a mixed profile constrained by conditions in Proposition 9 is an equilibrium profile also in a finite Poisson game.

Recall that the numbers of L-voters and R-voters are Poisson random variables with means n_L and n_R, where $p := \frac{n_L}{n_L+n_R} > \frac{1}{2}$. With the mixing profile identified in Proposition 9, we have three types of voters. Random variable $k_{L,A}$ denotes the number of L-voters who vote sincerely $v_L = (1,1,0,-1)$ and random variable $k_{L,B}$ denotes the number of L-voters who vote strategically $v_L = (1,1,-1,0)$. By decomposition property of Poisson distribution, the two variable have Poisson distribution with means αn_L and $(1-\alpha)n_L$.

Tie 1 characterized by $S_{L_1} = S_{R_0}$ is an event $2k_{L,A} + k_{L,B} = k_R$, and Tie 2 characterized by $S_{L_0} = S_{R_1}$ is an event $k_{L,A} + 2k_{L,B} = 2k_R$. The chief problem is that random variables on the sides of the equations are not Poisson, because they are sum of *correlated* Poisson variables. We will deal with this problem by trying to use a close Poisson approximation to the variables. The key problem is to obtain a reasonable Poisson approximation to the distribution of $s = 2k_R$. The probability mass function of s is

$$f_s(s) = \begin{cases} 0 & \text{if } \frac{s}{2} > \lfloor \frac{s}{2}, \rfloor \\ \frac{e^{-n_R} n_R^{\frac{s}{2}}}{\frac{s}{2}!} & \text{if } \frac{s}{2} = \lfloor \frac{s}{2} \rfloor. \end{cases}$$

Thus, there is a difference between odd and even realizations. We suppress the differences by considering a random variable s' with the probability mass function:

$$f_{s'}(s') = \frac{1}{2}\frac{e^{-n_R} n_R^{\lfloor \frac{s'}{2} \rfloor}}{\lfloor \frac{s'}{2} \rfloor !}$$

The distribution functions of s and s' are very close to each other for a large n_R, and the expected values are identical. The idea of replacing s by s' is that we are ultimately interested the probability of a tie which is a limit of a series S_s of a sequence of probabilities of specific ties, $\lim_{s\to\infty} S_s$. Each series S_s is a sum two series $S_s^{odd} + S_s^{even}$, where $S_s^{odd} = 0$ for any s. Now, each specific tie at s is a joint realization of two independent random variables with probability mass functions $f_s(s)$ and $g_s(s)$. Consider an even s. By replacement, we are replacing a series S_s by a modified series where each sum of a pair of elements in an original series

$$f_s(s)g_s(s) + \underbrace{f_s(s+1)g_s(s+1)}_{0} = \frac{f_s(s)}{2}[g_s(s) + g_s(s)]$$

is changed into a sum of a new pair

$$\frac{f_s(s)}{2}[g_s(s) + g_s(s+1)] = f_{s'}(s)g_s(s) + f_{s'}(s+1)g_s(s+1).$$

Finally, we approximate s' by a Poisson random variable s'' with mean $2n_R$:

$$f_{s''}(s'') = \frac{e^{-2n_R}(2n_R)^{s''}}{s''!}$$

This approximation is the closest Poisson approximation since the random variables s' and s'' have an equal expected value. Now, the approximated value of Tie 1 is nominal value $V+1$ times probability of a difference of two Poisson variables with means $2\alpha n_L + (1-\alpha)n_L = (1+\alpha)n_L$ and n_R. This is given by density of Skellam distribution at zero,

$$e^{-[(1+\alpha)n_L + n_R]} 2\sqrt{(1+\alpha)n_L n_R}(V+1).$$

Similarly, the approximated value of Ties 2 is nominal value $V-1$ times probability of a difference of two Poisson variables with means $\alpha n_L + 2(1-\alpha)n_L = (2-\alpha)n_L$ and $2n_R$:

$$e^{-[(2-\alpha)n_L + 2n_R]} 2\sqrt{2(2-\alpha)n_L n_R}(V-1).$$

Ties 1 and 2 are equally valuable for L-voters, and α is the equilibrium mixed strategy of L-voters, if α satisfies

$$L(\alpha, p) := e^{(1-2\alpha)n_L + n_R}\sqrt{\frac{1+\alpha}{2(2-\alpha)}} = e^{(1-2p\alpha)(n_L+n_R)}\sqrt{\frac{1+\alpha}{2(2-\alpha)}} = \frac{V-1}{V+1}. \qquad (12)$$

The left-hand side $L(\alpha, p)$ in (12) is decreasing in α for any p, which is consistent with the intuition about stability of the mixed best response: If $L(\alpha, p) < \frac{V-1}{V+1}$, then Tie 2 (L-majority event) is more valuable than Tie 1 (L-minority event), and sincere L-voters switch to strategic voting. As a result, α decreases and $L(\alpha, p)$ increases. Vice versa, if $L(\alpha, p) > \frac{V-1}{V+1}$, then Tie 1 is more valuable than Tie 2, and strategic L-voters switch to sincere voting and increase α. To derive the equilibrium value of α^*, we can first see that $L(\frac{1}{2}, p)$ primarily depends on $e^-(n_L + n_R)$, and this term can be made arbitrarily close to zero for a large number of voters. In other words, for a sufficient number of voters, we know that L-voters mix more in favor of the sincere ballot, $\alpha^* > \frac{1}{2}$. If $L(1, p) > \frac{V-1}{V+1}$, (12) leads us to expect a sincere profile, $\alpha^* = 1$.[17]

7 Conclusions

In this paper, we have identified the equilibrium voting outcomes for alternative scoring electoral rules in a stylized electoral situation with a single ideological dimension, four generic types of candidates, and two randomly sized groups of rational voters who vote primarily to promote their preferred ideologies, and secondarily to support high-quality candidates. We have compared the outcomes in single-member and two-member districts for scoring rules that differ in the numbers of positive and negative votes.

Our electoral situation predicts that plurality-at-large almost always ends in a non-mixed outcome where two candidates of the same group gain both seats. This prediction corresponds to 80% of outcomes of the recent elections into the assemblies of the U.S. states that actually employ plurality-at-large in double-member district. By adding unrestricted numbers of votes (approval voting with an arbitrary number of positive votes or combined approval-disapproval voting with an arbitrary number of positive and negative votes), the strict equilibria only replicate the non-mixed outcomes.

In contrast, the 2+1 rule yields a strict and sincere equilibrium with a large frequency of mixed outcomes, hence protects interests of the second largest group relative more than alternative two-seat electoral rules. For sufficiently symmetric distributions, the 2+1 rule motivates the voters to cast the negative vote only to the low-quality candidate from the opponent group. Two positive votes are thus cast along the more salient ideological dimension, and a single negative vote is cast along the less salient quality dimension.

The beneficial effects of the 2+1 rule are related to the existence of two types of pivotal events for any group: In one event, the group is in minority and competes for the last seat with her stronger candidate. In the other event, the group is in majority and competes for the last seat with her weaker candidate. Each event contains a different pair of contenders. A voter cannot cumulate positive votes to a single preferred candidate, hence allocates positive votes to both candidates in order to influence both events. At the same time, a voter picks up only one of the events when allocating the single negative vote. Non-substitutability of the positive votes combined with a single negative vote preserves the desirable multiplicity of the pivotal events.

[17]Notice however that (12) contains two approximations, hence the value $L(1, p)$ is not the exact existence condition for the sincere equilibrium.

The main result is constructed for a policy benefit from a seat being independent on the number of seats gained. In fact, winning the first seat may be marginally more valuable then winning the second seat, and the policy benefit may decrease in the number of seats won by the group. This effect would make the minority events even more valuable, hence would reinforce our equilibrium with the negative votes to the low-quality candidates.

Although the main purpose of the paper is to compare the 2+1 electoral rule with plurality-at-large and the closest alternatives in double-member districts, a more general contribution of the paper is into understanding of how scoring electoral rules in majoritarian multi-member districts affect voters' tradeoffs over valent and conflicting political issues. Adding an extra negative vote has a different effect than adding an extra positive vote because of a different effect on the expected scores of serious candidates and non-serious candidates, and because of the restraint to cumulation of points to a single candidate.

We find that simultaneous presence of negative and positive votes increases the voters' scope for discrimination. Nevertheless, an effectively discriminative mix of positive and negative votes must avoid the adverse underdog effect of an excessively large set of viable candidates and the incentive to overstate that results in the serious candidates receiving only points $-1, 1$, not points $-1, 0, 1$. The 2+1 rule achieves that goal by limiting the number of negative votes relative to the number of the positive votes.

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A Proofs

A.1 Proof of Proposition 1 (Plurality)

W.l.o.g., we examine deviations of L-voter in the set of admissible ballots:

- With all votes inactive, $v_L = (0,0,0,0)$, all candidates are serious for every k, and L-voter deviates by $v_L = (1,0,0,0)$.
- Vote for the 1st candidate, $v_L = (1,0,0,0)$: There is a single tie $k = \frac{n}{2}$. In all pivotal events, L_1 and R_1 compete for the seat. L_0 and R_0 are not serious candidates. Voter L's pivotal events are $\{\frac{n}{2}, \frac{n}{2}+1\}$ and R's pivotal events are $\{\frac{n}{2}-1, \frac{n}{2}\}$. L-voter may only lose by any deviation, hence this profile is an equilibrium.
- Vote for the 2nd candidate, $v_L = (0,1,0,0)$: There is a single tie $k = \frac{n}{2}$. In all pivotal events, L_0 and R_0 compete for the seat. L_1 and R_1 are not serious candidates. L-voter may only lose by deviation, hence this profile is also an equilibrium.
- Vote for the 3rd candidate, $v_L = (0,0,1,0)$: There is a single tie $k = \frac{n}{2}$. In all pivotal events, L_1 and R_1 compete for the seat. L-voter deviates by $v_L = (1,0,0,0)$. □

A.2 Proof of Proposition 2 (Mixed)

Since 1+1 rule contains weakly more votes of any type than 1+0 rule, any deviation present in 1+0 rule is also a deviation in the 1+1 rule. This principle will be used in other proofs as well. When using this principle, however, bear in mind that M-seat ties count, hence M must be constant for applicability of the principle.

- For any equilibrium profile from the 1+0 rule, L-voter deviates by using his or her negative vote. The negative vote changes the near tie $k = \frac{n}{2} - 1$ into a tie and wins tie $k = \frac{n}{2}$.

- Positive vote inactive, negative vote active: Two candidates A and B (those not receiving negative votes) are serious for every k. If L-voter deviates by casting a positive vote to the preferred candidate out of A, B, he or she wins all realizations $k = 1, \dots, n$ (but obviously not the realization $k = 0$). This is a strict improvement for $k = 1, \dots, n-1$. For $k = n$, there may be a loss if the candidate D who is serious in the event n is very valuable. However, for a sufficiently large n, the discrete loss in a single realization is always compensated by the sum of discrete gains in $n-1$ realizations.

We are left with those profiles where all votes are active. We first rule out profiles where a pair of candidates A, B receives positive votes and a pair C, D receives negative votes: There is a single tie $k = \frac{n}{2}$. In all pivotal events, A and B compete for the seat. C and D are not serious candidates. L-voter deviates by transferring the negative vote to the worse of the candidates A and B.

Thus, only two candidates receive both positive and negative votes. Thus, these votes cancel out in tie $k = \frac{n}{2}$, where $S_c(k) = 0$ for any $c \in K$. In the tie, the expected payoff is $\frac{2V+2}{4} = \frac{V+1}{2}$.

- $v_L = (0, 1, 0, -1)$: L-voter deviates to $v_L = (1, 0, 0, -1)$. Thereby, L-voter wins tie $k = \frac{n}{2}$ with the 1st candidate L_1 (gain). Also, L-voter changes win of L_0 at near tie $k = \frac{n}{2} + 1$ into a ties between L_1 and L_0 (gain). The elected candidate for $k < \frac{n}{2}$ and $k > \frac{n}{1} + 1$ is not changed. This is not an equilibrium.

- $v_L = (1, 0, -1, 0)$: L-voter considers deviating to $v_L = (0, 1, -1, 0)$ (positive vote). Thereby, L-voter replaces tie of four candidates at $k = \frac{n}{2}$ with the win for the 2nd candidate L_0 (gain $V - \frac{V+1}{2} = \frac{V-1}{2} > 0$). But, at the same time, L-voter changes the win of L_1 at near tie $k = \frac{n}{2} + 1$ into a tie between L_1 and L_0 (a loss $-\frac{1}{2}$). The elected candidates for $k < \frac{n}{2}$ and $k > \frac{n}{1} + 1$ are not changed. The deviation does not make L-voter worse off iff $b_L(\frac{n}{2})(\frac{V-1}{2}) - b_L(\frac{n}{2} + 1)\frac{1}{2} \geqslant 0$. We use that $b_L(\frac{n}{2}) = b_L(\frac{n}{2} + 1)$, hence the condition rewrites into $V \geqslant 2$.

  L-voter may consider deviation to $v_L = (1, 0, 0, -1)$ (negative vote). The only effect is that R_1 wins $k = \frac{n}{2}$ with a loss $1 - \frac{V+1}{2} = \frac{1-V}{2} < 0$.

  Also, L-voter may consider deviation to $v_L = (0, 1, 0, -1)$ (positive and negative vote). At $k = \frac{n}{2}$, there is now a tie of R_1 and L_0 with exactly zero gain. At $k = \frac{n}{2} + 1$, there is now a tie of R_1 and L_1 which means a loss $-\frac{V}{2} < 0$. Finally, it is easy to see that there is no better deviation than the one considered up to now. □

A.3 Proof of Proposition 3 (3+0, $M = 1$)

First of all, we consider equilibrium profiles from the 1+0 rule: For Pareto-inferior profile, we use that ballot $v_L = (0, 1, 0, 0)$ is weakly dominated by ballot $v_L = (1, 1, 0, 0)$. For Pareto-superior profile, only two candidates are serious. L-voter supports the better of the two serious candidates. Thus, L-voter cannot add extra positive vote to improve his or her payoff. Also, the ballot $v_L = (1, 0, 0, 0)$ is not weakly dominated by any other ballot since in alternative profiles, there might be a tie between the 1st and 2nd (or 3rd) candidate, and casting the additional positive vote may imply a utility loss in the tie. The Pareto-superior profile remains the equilibrium profile.

The next set of profiles involve two positive votes:

- Votes for the 1st and 3rd candidates, $v_L = (1, 0, 1, 0)$: In any k, there is a tie between L_1 and R_1. L-voter deviates by withdrawing the positive vote for R_1.
- Votes for two different pairs of candidates: In any $k \neq \frac{n}{2}$, there is a tie within the pair of preferred candidates. Suppose L-voter prefers candidates A and B, and $u_L(A) > u_L(B)$. Then, L-voter deviates by withdrawing the positive vote for B. This implies a gain for $k = \frac{n}{2} + 1, \ldots, n$ and a potential loss at $k = \frac{n}{2}$. For a sufficiently large n, the discrete loss in a single realization is always compensated by discrete gains in $\frac{n}{2} - 1$ realizations.

Finally, consider all three votes to be cast, $v_L = (1, 1, 1, 0)$. Then, the configuration is identical to the 1+1 rule with only one negative vote active. Two candidates A and B (those not receiving negative votes) are serious for every k. If L-voter deviates by casting a positive vote to the preferred candidate out of A, B, he or she wins all realizations $k = 1, \ldots, n$ (but obviously not the realization $k = 0$). This is a strict improvement for $k = 1, \ldots, n - 1$. For $k = n$, there may be a loss if the candidate D who is serious in the event n is very valuable. However, for a sufficiently large n, the discrete loss in a single realization is always compensated by the sum of discrete gains in $n - 1$ realizations. □

A.4 Proof of Proposition 4 (2+0)

W.l.o.g., we examine deviations of L-voter in the set of admissible ballots. We prove that all votes must be active:

- With all votes inactive, $v_L = (0, 0, 0, 0)$, all candidates are serious for every k, and L-voter deviates by $v_L = (1, 1, 0, 0)$.
- Vote for a single candidate A. For L-voter, there is only a single tie $k = n$ of candidates B, C and D. (Recall that tie at $k = 0$ is not relevant for L-voter, since the event $k = 0$ involves only a set of R-voters.) L-voter deviates by casting the extra positive vote to the best of the candidates B, C and D.

Three pairs of candidates may receive two positive votes:

- Vote for the 1st and 2nd candidates, $v_L = (1, 1, 0, 0)$: There is a single tie $k = \frac{n}{2}$, where all four candidates compete for the seat. At tie, any deviation makes L-voter strictly worse off. At near tie $k = \frac{n}{2} + 1$, a transfer of a positive vote from own candidate (L_1 or L_0) to any other candidate (R_1 or R_0) induces a tie, and this makes L-voter strictly worse off. Thus, this is a strict equilibrium.

- Vote for the 1st and 3rd candidates, $v_L = (1, 0, 1, 0)$: For any k, $S_{L_1}(k) = S_{R_1}(k) = n > 0 = S_{L_0}(k) = S_{R_0}(k)$. There is no pivotal event and all candidates are non-serious. This is a weak equilibrium.
- Vote for the 2nd and 3rd candidates, $v_L = (0, 1, 1, 0)$: There is a single tie $k = \frac{n}{2}$, where all four candidates compete for the seat. At tie, L-voter becomes better off by transferring a positive vote from the 3rd candidate R_1 to the 1st candidate L_1. $\square$

A.5 Proof of Proposition 5 (2+1)

Since the 2+1 rule contains weakly more votes of any type than 2+0 rule, any deviation present in the 2+0 rule is also a deviation in the 2+1 rule. We investigate if the two equilibrium profiles from the 2+0 rule remain in the equilibrium for the 3+0 rule:

- Votes for the 1st and 2nd candidates, $v_L = (1, 1, 0, 0)$: There is a single tie $k = \frac{n}{2}$, where all four candidates compete for the seat. Consider now L-voter's deviation to $v_L = (1, 1, 0, -1)$. This strictly improves payoff at tie $k = \frac{n}{2}$ and does not affect the probabilities of being elected for any $k \neq \frac{n}{2}$.
- Votes for the 1st and 3rd candidates, $v_L = (1, 0, 1, 0)$: There is no pivotal event and all candidates are non-serious. This is a weak equilibrium but not in admissible strategies. The ballot $v_L = (1, 0, 1, 0)$ is weakly dominated by a ballot $v_L = (1, 0, 1, -1)$.

Consider now only a single negative vote being active. All such profiles involves ties for any k and all candidates are serious. L-voter deviates by casting positive votes to 1st and 2nd candidates and strictly improves his or her expected payoff. The main reason for improvement is that for any k, a positive vote to the 2nd candidate cannot reduce probability of the 1st candidate to be elected, given that also the 1st candidate now obtains a positive vote.

Consider a positive and a negative vote being active.

- A pair of candidates A, B receives positive votes and a pair C, D receives negative votes. There are ties at extreme realizations, $k \in \{0, n\}$. For $k = 0$, A competes with C and the other candidates are not serious in the event $k = 0$. For $k = n$, B competes with D and the other candidates are not serious in the event $k = n$. L-voter cannot change the tie $k = 0$, but affects the tie $k = n$. Therein, L-voter can be made strictly better off by deviation. If $u_L(B) > u_L(D)$, then L-voter adds a positive vote to B. If $u_L(B) < u_L(D)$, then L-voter transfers the negative vote from C to D. Both deviations affect only the realization $k = n$ and each is a strict improvement.

- A pair of candidates A, B receives both positive and negative votes. Suppose the pair is (L_1, R_1), hence $v_L = (1, 0, -1, 0)$. There are ties for any k where L_0 is always a serious candidate. L-voter deviates by adding a positive vote to L_0, hence $v_L = (1, 1, -1, 0)$. For any $k \neq \frac{n}{2}$, L_0 now wins a seat in the competition win R_0. For $k = \frac{n}{2}$, L_0 now wins the first seat and the other three candidates compete for the second seat. This is also a strict improvement, because $V + \frac{V+2}{3} > \frac{V+1}{2}$.

- A pair of candidates A, B receives both positive and negative votes. Suppose the pair is (L_0, R_0), hence $v_L = (0, 1, 0, -1)$. There are ties for any k where L_1 is always a serious candidate. L-voter deviates by adding a positive vote to L_1, hence $v_L = (1, 1, 0, -1)$. This is clearly an improvement for any k.

The remaining profiles are for all votes being active. Clearly, we eliminate profiles where a positive and negative vote from one voter is to the same candidate, because this would be equivalent to having only a negative vote active. (The same incentives to deviate would apply because the voter would face the same strategy set once he withdraws both the positive and negative vote from the ballot.) We are left with five profiles.

- Ballot $v_L = (1, -1, 1, 0)$. There is no pivotal event and all candidates are non-serious. This is a weak equilibrium but not in admissible strategies.

- Ballot $v_L = (1, 0, 1, -1)$. There is no pivotal event and all candidates are non-serious. This is a weak equilibrium in admissible strategies.

- Ballot $v_L = (1, 1, 0, -1)$. The vector of score functions is $(k, 2k-n, n-k, n-2k)$. There are two ties, $k_A = \frac{n}{3}$ and $k_B = \frac{n}{3}$. In the tie k_A, L_1 and R_0 are serious. In the tie k_B, L_0 and R_1 are serious. L-voter cannot transfer any positive vote to gain for some k. The negative vote can be transferred from R_0 to R_1. Then, R_0 wins a seat at k_A against L_1 instead of a tie (a loss $\frac{-(V+1)}{2} < 0$) and R_1 loses a seat at k_B against L_0 instead of a tie (a gain $\frac{V-1}{2} > 0$). Now, we use that $b_L(k_B) = 2b_L(k_A)$. The expected gain is not positive if and only if

$$b_L(k_A)\left(-\frac{V+1}{2} + V - 1\right) \leqslant 0,$$

  which is equivalent to $V \leqslant 3$. Under this condition, this sincere profile is an equilibrium.

- Ballot $v_L = (1, 1, -1, 0)$. The vector of score functions is $(2k-n, k, n-2k, n-k)$. There are two ties, $k_A = \frac{n}{3}$ and $k_B = \frac{2n}{3}$. In the tie k_A, L_0 and R_1 are serious. In the tie k_B, L_1 and R_0 are serious. L-voter cannot transfer any positive vote to realize gains for any k. The negative vote can be transferred from R_1 to R_0. Then, R_1 wins a seat at k_A against L_0 instead of a tie (a loss $\frac{-(V-1)}{2} < 0$) and R_0 loses a seat at k_B against L_1

instead of a tie (a gain $\frac{V+1}{2} > 0$). Again, we use $b_L(k_B) = 2b_L(k_A)$. The expected gain is always positive,

$$b(k_A)\left(-\frac{V-1}{2} + V + 1\right) = \frac{b(k_A)}{2}(V+3) > 0.$$

- Ballot $v_L = (0, 1, 1, -1)$. The vector of score functions is $(k, n-2k, n-k, 2k-n)$. There are two ties, $k_A = \frac{n}{3}$ and $k_B = \frac{n}{3}$. In the tie k_A, L_0 and L_1 are serious. In the tie k_B, R_1 and R_0 are serious. L-voter deviates by transferring a positive vote from L_0 to L_1. Then, L_1 wins a seat at k_A against L_0 instead of a tie (a gain $\frac{1}{2} > 0$). □

A.6 Proof of Proposition 6 (3+0, $M = 2$)

Since 3+0 rule contains weakly more votes of any type than 2+0 rule, any deviation present in the 2+0 rule is also a deviation in the 3+0 rule. We investigate if the two equilibrium profiles from the 2+0 rule remain in the equilibrium for the 3+0 rule:

- Votes for the 1st and 2nd candidates, $v_L = (1, 1, 0, 0)$: There is a single tie $k = \frac{n}{2}$, where all four candidates compete for the seat and the expected payoff from both seats is $V + 1$. Extension of the strategy set by approval voting means that at the tie, L-voter may now considers deviation to $v_L = (1, 1, 1, 0)$. This wins a single seat to R_1 at $k = \frac{n}{2}$ and the payoff from the first seat is 1. There is a tie for the second seat for the remaining candidates, and the expected payoff from the second seat is $\frac{2V+1}{3}$. This deviation makes L-voter worse off because $1 + \frac{2V+1}{3} < V + 1$ is equivalent to $1 < V$. At the same time, this deviation has no effect on the probabilities of being elected at any $k \neq \frac{n}{2}$. Thus, this remains as an equilibrium.

- Votes for the 1st and 3rd candidates, $v_L = (1, 0, 1, 0)$: There is no pivotal event and all candidates are non-serious. This is again a weak equilibrium.

Additionally, consider all three votes to be cast, $v_L = (1, 1, 1, 0)$. In contrast to $M = 1$, ties for the second seat are only for the extreme realizations $k \in \{0, n\}$. Consider now L's deviation to $v_L = (1, 1, 0, 0)$ (a vote for the 3rd candidate is withdrawn). We have effects at three realizations:

- At tie $k = n$, the two seats are now won by L_1 and L_0, and the payoff from the two candidates is $2V + 1$. The expected payoff without deviation was $\frac{4}{3}(V + 1)$, hence there is a gain $2V + 1 - \frac{4}{3}(V + 1) = \frac{2V-1}{3} > 0$.

- At near tie $k = n - 1$, there is now a tie over the second between R_1 and L_0, hence the expected payoff is $V + 1 + \frac{V+1}{2}$. The expected payoff without deviation was $V + 2$, hence there is a gain $\frac{V+1}{2} - 1 = \frac{V-1}{2} > 0$.

- At near tie $k = 1$, L_1 gains the first seat and R_1 and R_0 now compete for the second seat. The payoff is $V + 1 + \frac{1}{2}$. The expected payoff without deviation was $V + 2$, hence there is a loss $-\frac{1}{2} < 0$.

Since $b_L(1) = b_L(n)$ and $b_L(n-1) = (n-1)b_L(1)$, the deviation does not make L-voter worse off iff

$$b(1)\left(\frac{2V-1}{3} - \frac{1}{2}\right) + (n-1)b(1)\left(\frac{V-1}{2}\right) \geqslant 0. \tag{13}$$

The condition rewrites into $V(3n+1) - (3n+2) \geqslant 0$, or equivalently, $V \geqslant \frac{3n+2}{3n+1}$. Finally, see that there is no better deviation than the one considered up to now. □

A.7 Proof of Proposition 7 (3+3, $M = 2$)

The 3+3 rule admits all ballots under 2+1 and 3+0 rules, hence we first verify stability of the equilibria for these rules. For approval voting, we rule out all profiles on the grounds of inadmissibility; by admissibility, the worst candidate must receive a negative vote. For profiles identified by the 2+1 rule:

- $v_L = (1, 0, 1, -1)$ (passive equilibrium for 2+1): None event is serious, hence this remains a passive equilibrium.
- $v_L = (1, 1, 0, -1)$ (strict equilibrium for 2+1): Each voter adds an extra negative vote to the high-quality candidate of the other group in order to change the winner in her less valuable pivotal event.

We examine all extra admissible profiles relative to 2+1 rule. We begin with those that admit two or three negative votes which was not feasible under 2+1:

- $v_L = (1, -1, -1, -1)$: For L-voter, consider tie $k = n$. By changing the score for L_0 candidate into $v_L^{L_0} = 1$, L_0 wins this tie against R_1 and R_0, and no other effect takes place.
- $v_L = (1, 1, -1, -1)$: There is a single tie $k = \frac{n}{2}$, where all four candidates compete for the seat. At tie, any deviation makes L-voter *strictly* worse off. At near tie $k = \frac{n}{2} + 1$, any decrease in points of own candidates (L_1 or L_0) and/or any increase in points of the other candidates (R_1 or R_0) cannot make L-voter better. Thus, this is a strict equilibrium.
- $v_L = (1, -1, 1, -1)$: There is no pivotal event since $S_{L_1}(k) = S_{R_1}(k) = n > -n = S_{L_0}(k) = S_{R_0}(k)$. Hence, this is a passive equilibrium.

- $v_L = (1, 0, -1, -1)$: This profile is similar to the strict profile under 2+1 rule. There are two ties, $k \in \{\frac{n}{3}, \frac{2n}{3}\}$, each with a pair of serious candidates (R_1, L_0) and (R_0, L_1). Each voter deviates by adding an extra positive vote to the low-quality candidate of own group to change the winner in her more valuable pivotal event.

- $v_L = (1, -1, 0, -1)$: There is no pivotal event since $S_{L_1}(k) = S_{R_1}(k) \geqslant 0 > -n = S_{L_0}(k) = S_{R_0}(k)$. Hence, this is a passive equilibrium.

Finally, we examine profiles that involve negative votes (unlike 3+0), and have more than two positive votes (unlike 2+1). This is a single admissible profile:

- $v_L = (1, 1, 1, -1)$: L-voter deviates to $v_L = (1, 1, 0, -1)$. At tie $k = n$, there is a gain of having (L_1, L_0) among the winners instead of (L_1, L_0, R_1). At near ties $k = n - 1$ and $k = 1$, there are no effects on the sets of winning candidates. □

A.8 Proof of Proposition 8 (Welfare)

By Proposition 5, the sufficient and necessary condition for a strict equilibrium under the 2+1 rule is $V \leqslant 3$. For $V \leqslant 3$, $\hat{\phi} \leqslant \frac{1}{3}$. Since $\hat{\phi} \leqslant \frac{1}{3}$, both mixed and non-mixed outcomes disproportionately favor R_0 to L_1. However, the distortion of the mixed outcome O_m occurs in the interval $\phi \in [\hat{\phi}, \frac{1}{3}]$ which is a proper subinterval of $\phi \in [\hat{\phi}, \frac{1}{2}]$ where the non-mixed outcome O_n distorts. Hence, under $V \leqslant 3$, $W(O_m) > W(O_n)$. □

A.9 Proof of Proposition 9 (2+1, asymmetry)

The proof is in three steps.

1. First, we prove that sincere profile is not an equilibrium. There are two ties, $k_A = \frac{n}{3}$ and $k_B = \frac{2n}{3}$. From Proof to Proposition 5, the expected gain of L-voter from transferring a negative vote from R_0 to R_1 is positive, $-b_L(k_A)\frac{V-1}{2} + b_L(k_B)\frac{V+1}{2} > 0$.

2. Second, we prove that L-voters do not play a pure strategy of negative vote against R_1. In such a profile, the expected gain of L-voter from transferring a negative vote from R_1 to R_0 is positive, $-b_L(k_A)\frac{V}{2} + b_L(k_B)\frac{V}{2} > 0$.

3. Third, we can prove why L-voter mixes asymmetrically to R_1 and R_0. By contradiction, suppose L-voter mixes symmetrically. Let $\pi(R_1, L_1) = \Pr(S_{R_0} > S_{R_1} = S_{L_1} > S_{L_0})$ be the probability of the pivotal event of R_1 against L_1. By mixing symmetry, $\pi_0 := \pi(R_1, L_1) = \pi(R_0, L_1)$. Let $\pi(R_1, L_0) := \Pr(S_{L_1} > S_{L_0} = S_{R_1} > S_{R_0})$ be the probability of the pivotal event of R_1 against L_0. By mixing symmetry, $\pi_1 := \pi(R_1, L_0) = \pi(R_0, L_0)$.

   Payoff-equalizing property is violated, because the negative vote against R_1 brings less than the negative vote against R_0,

$$\pi_0 V + \pi_1(V - 1) < \pi_0(V + 1) + \pi_1 V.$$

Thus, L-voter deviates to mixing against R_0 if profile is in symmetric mixing. Thus, the equilibrium contains mixing against R_1, but with a lower probability than mixing against R_0. $\square$

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