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DUOPOLISTIC COMPETITION ON A PLANE

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Duopolistic Competition On a Plane

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Abstract:
The paper models duopolistic competition in so called monotowns: towns with one big factory where most of the citizens are employed. Workers after job go to one of the competitor shops, buy the product and bring it home. Nash equilibrium is found for linear and two-dimensional cases. The principle of maximum differentiation is refuted.

JEL: C72, D43, L13, R32
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1 Introduction

Modelling agglomeration process is a problem of high interest and models of spatial competition describe this process most visually. In a big number of papers written after seminal Hotelling [1929] spatial competition was considered for one dimensional case which is less applicable to the real world situations. That is why I have decided to investigate two dimensional cases. The real world material which backs up my model are so-called monotowns: "Monotowns are urban settlements with economic bases dominated by a single industry [...] The Ministry of Economy, classified 467 cities and 332 smaller towns as monotowns roughly two in five of Russias cities" The World Bank in Russia [2010]. There are such kind of cities in the USA as well, for example in Everett in Washington 39,000 employees out of total 64,300 are working at The Boeing Company (*Comprehensive Annual Financial Report [2012]*). In this paper I consider duopolistic competition model in a two dimensional city of convex geometric shape, namely, a disk and a square with two firms selling identical products competing with each other. There is a factory in the city where all consumers are employed. A consumer first should go to the factory and then go to one of the firms, having bought one unit of the product, she goes back home. So the factory is a point of attraction in the city. Consumers are minimizing their costs and firms are maximizing their profits. Firms should first choose their locations simultaneously and then set prices. Thus, we are looking for the location-price equilibria of the two-stage noncooporative game with complete information. In essence, it is an extension of the Hotelling [1929] classical model in which he considered the linear city. But as two dimensional models are more applicable to the real world cases, modelling spatial competition on a plane is of higher interest and less investigated because of its difficulty, than linear models. For example Tharakan&Thisse [2002] have applied Hotelling linear duopoly model to the international trade where countries are linear, stating that: "we want to account for the fact that countries have different geographical sizes while consumers are dispersed over the corresponding areas. In such a setting, it becomes possible to study the implications of trade according to the relative position of firms and consumers within the integrated space, a task that cannot be accomplished in standard trade models in which countries are dimensionless." Of course for modelling international trade it would have been more appropriate to consider countries being two-dimensional instead of linear. This example demonstrates how important it is for Economic theory to investigate two dimensional spatial competition and no doubt that transportation costs play the crucial role in international trade.

While two dimensional cases of Hotelling's model were considered by number of authors (among them are Tabuchi [1994], Mazalov&Sakaguchi [2003], Shchiptsova [2012]), I decided to modify one of the latest models, namely, the model from Shchiptsova [2012]. In particular, I generalize the model by allowing firms to locate broader than just on diameter (I allow firms to locate on any chord) and I also change the consumer cost function taking into account the weight of the product. Surprisingly none of authors took into account the weight of product yet. The city in Shchiptsova [2012]'s model is a unit disk, but I consider also the case when the city has a shape of a square like in Tabuchi [1994] and Irmen&Thisse [1998]. As we know some cities look like disks and some resemble squares.

In his seminal paper Hotelling [1929] considered the following model: in a [0,1] linear segment city where consumers are distributed uniformly, two competitor firms producing the identical products with zero marginal cost need to choose their locations and prices of product. The consumer cost function was considered as a linear function of the product price and the distance from a consumer to a firm. A consumer chooses the firm buying from which is less costly. Hotelling [1929] showed that in this case both firms will locate at the center of the city. This result is known as a *principle of minimum differentiation.* Hotelling noted that his model (without prices) might be applied to political competition as well stating that "each party strives to make its platform as much like the other's as possible". The other famous paper Black [1948] based on the Hotelling linear duopoly model is a benchmark one in the political economy theory. Black has derived well-known *'median voter theorem'* using the same logic as in location competition in Hotelling duoploy: in Nash equilibrium two competitor political parties choose the central political positions (the position of the *median* voter).

Since 1929, Hotelling's model has been revised in a number of papers. The benchmark one among them is paper by d'Aspremont et al. [1979] in which it was shown that Hotelling's *principle of minimum differentiation* is invalid because when firms are located sufficiently close to each other, there exists no pure strategy price equilibrium of the game. To fix Hotelling's imperfection d'Aspremont et al. [1979] modified his model by changing consumer cost function from linear to quadratic. As a result, they showed that in this case price equilibrium exists for any location of firms. Moreover, in d'Aspremont et al. [1979] was obtained the result opposite to Hotelling's one, namely, it was shown that competitor firms tend to allocate at the endpoints of the linear city, i.e. they maximize differentiation. Economides [1986] had shown that if the transportation cost function is a power function of power $\frac{5}{3} \leq \alpha \leq 2$ then in equilibrium firms locate at the extreme points of the linear segment city, while if $\frac{5}{3} \geq \alpha \geq 1.26$ equilibrium points are "strictly inferior to [0,1]". It should be mentioned that Hotelling's result is valid for location game and so it could be applied to political competition.

Bester et al. [1996] noticed the following coordination problem in the solution proposed in d'Aspremont et al. [1979]: in the equilibrium one firm should locate at the one end of the city and the second one at the other end;

but as the game is noncooperative it can happen that both firms locate at the same end, which will lead to setting the prices equal to zero. In order to avoid such a undesirable scenario Bester et al. [1996] suggested *a mixed-strategy location equilibrium*, which is that one of firms mixes choosing between two endpoints with equal probabilities 1/2 and the second firm locates at the center of the city. In Bester et al. [1996], it was also shown that there exist infinitely many mixed-strategy equilibria in this game.

Hotelling's linear city model was further generalized for two dimensional cases which are more applicable to the real world problems. Tabuchi [1994] has considered location-price game for rectangular city, Irmen&Thisse [1998] have considered location-game in hypercube in both papers consumer cost function is quadratic a la d'Aspremont et al[1979]. A noteworthy model of two dimensional city was designed by Mazalov&Sakaguchi [2003] who considered a city as a unit disk and found the pure strategy Nash equilibrium for the noncooperative location-price game with quadratic consumer cost function. For the case of uniformly distributed consumers Mazalov&Sakaguchi [2003] showed that firms should maximize differentiation like in one dimensional case. Stadler [2017] allowing firms to locate outside the disk city as well, asserted that rivals should locate $x_1 = -x_2 = 1.178$ if the consumers are distributed uniformly. Gao er al. [2014] have applied the discrete (when consumers are located at the nodes - cities with airports of the graph - airline) model of two dimensional location-price game with many players to airline markets and found equilibria for Russian and Chinese airline networks.

A novel modification of Mazalov's spatial competition model was proposed by Shchiptsova [2012] who considered the market with two producers (competitors) of one type product and the third producer (monopolist) of a different type product; each consumer needs to buy one unit of product of both types; cost of manufacturing the product is zero. The position of the monopolist firm is given endogenously and competitor firms are allowed to locate only on the diameter going through the point where the monopolist is located. The consumer's cost function is quadratic with respect to distance. Each consumer chooses a firm buying from which is less costly for her. The competitor firms first choose their locations simultaneously and then they set prices simultaneously. Shchiptsova [2012] found the Nash equilibrium of the game, where competitor firms should locate and what prices they should set given the location of the monopolist firm.

Since the price of the product manufactured by the third (monopolist) firm does not play any role in the above-mentioned model, instead of a monopolist firm I consider the factory (job) where all consumers are employed and after the job, each consumer chooses to which firm to go. Such a vision of the model is more realistic being based on the example of monotowns. The first question I am going to discuss in this paper is how the equilibrium in the Shchiptsova's model changes when we modify the cost function; namely

I assume that a consumer's traveling cost from firm to home after having bought the product is twice as greater as traveling without the product, i.e the modified consumer cost function will be as follows:

$$C_i = p_i + d^2(home, factory) + d^2(factory, firm_i) + 2d^2(firm_i, home) \quad i = 1, 2$$

The economic reasoning here is that product is heavy or bulky and taking it home is costly for a consumer.

The one of the other problems that I address in my work is what happens if there is only one street (chord) in the city on which competitor firms are allowed to locate (factory is not located on that street). This question I discuss for the square city as well. Finally, I return to classical linear case to find out what does the introduction of the third point (factory) in the d'Aspremont et al. [1979]'s model change and whether the *principle of maximum differentiation* still holds in the modified model.

2 Disk City

2.1 Overview of Shchiptsova's model

The following problem was addressed in Shchiptsova [2012]: in the unit circle with radius 1 there are two firms located at the points A_1 and A_2 that sell the product of the same type, whereas the third firm located at point B sells a product of a different type. Points A_1 and A_2 are restricted to be located on the diameter passing through the point B. Only the two firms selling the products of the same type compete between each other.

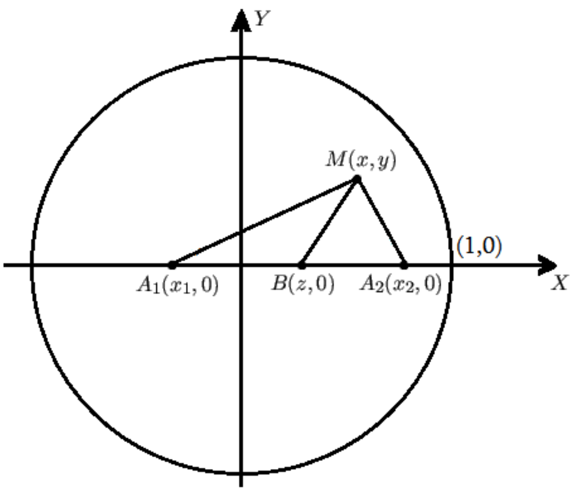

Figure 1: location on the diameter of the disk

The marginal cost of production is 0. Consumers in the circle are distributed uniformly. Each consumer consumes the products of both types. The cost function of a consumer located at an arbitrary point $M(x, y)$ is

$$C_i(x, y) = p_i + d^2(M, A_i) + p_3 + d^2(A_i, B) + d^2(B, M)$$

$i = 1, 2$ where d is the Euclidean distance. The profit of a competitor firm is $\Pi_i = p_i S_i$ $i = 1, 2$ where S_i is the market share of i-th firm so $S_1 = 1 - S_2$. The competitor firms should locate on the diameter that goes through the monopolist firm. Competitors are maximizing their profits. The game is non-cooperative with complete information and the competitor firms make their decisions in the following order:

1. Choose location simultaneously.
2. Choose prices simultaneously.

The aim is to find a Nash equilibrium.

Sketch of Shchiptsova [2012]'s solution: Assume that $x_1 \leq x_2$. The equation of vertical line dividing the circle into S_1 and S_2 is obtained from solving:

$$C_1(x, y) = C_2(x, y)$$

$p_1 + (x - x_1)^2 + y^2 + (x_1 - z)^2 + p_3 + (z - x)^2 + y^2 = p_2 + (x - x_2)^2 + y^2 + p_3 + (x_2 - z)^2 + (z - x)^2 + y^2$

$$\bar{x} = \frac{p_2 - p_1}{2(x_2 - x_1)} + x_1 + x_2 - z$$

In polar coordinates S_2 could be written as

$$S_2 = \int_{\bar{x}}^{1} \int_{-arccos\frac{x}{r}}^{arccos\frac{x}{r}} r f(\theta, r)\, d\theta dr$$

The firms first choose their locations so they solve the maximization problem

$$\begin{cases} \frac{d\Pi_1}{dx_1} = \frac{\partial p_1}{\partial x_1}(1 - S_2) - p_1(\frac{\partial S_2}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial S_2}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial S_2}{\partial x_1}) = 0 \\ \frac{d\Pi_2}{dx_2} = \frac{\partial p_2}{\partial x_2}S_2 + p_2(\frac{\partial S_2}{\partial p_1}\frac{\partial p_1}{\partial x_2} + \frac{\partial S_2}{\partial p_2}\frac{\partial S_2}{\partial x_2} + \frac{\partial S_2}{\partial x_2}) = 0 \end{cases} \quad (1)$$

When x_1 and x_2 are fixed firms maximize profits with regard to prices:

$$\begin{cases} \frac{\partial \Pi_1}{\partial p_1} = 1 - S_2 - p_1\frac{\partial S_2}{\partial p_1} = 0 \\ \frac{\partial \Pi_2}{\partial p_2} = S_2 + p_2\frac{\partial S_2}{\partial p_2} = 0 \end{cases} \quad (2)$$

Substituting S_2 from (2) in (1) we will obtain

$$\begin{cases} \frac{d\Pi_1}{dx_1} = -p_1(\frac{\partial S_2}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial S_2}{\partial x_1}) = 0 \\ \frac{d\Pi_2}{dx_2} = p_2(\frac{\partial S_2}{\partial p_2}\frac{\partial S_2}{\partial x_2} + \frac{\partial S_2}{\partial x_2)} = 0 \end{cases} \tag{3}$$

From these systems of equations the next system could be obtained

$$\begin{cases} (2\alpha + \frac{2p_1+p_2}{x_1-x_2})\frac{\partial S_2}{\partial p_2} + p_2\alpha\frac{\partial^2 S_2}{\partial p_2^2} = 0 \\ (2\beta + \frac{2p_2+p_1}{x_1-x_2})\frac{\partial S_2}{\partial p_2} - p_1\beta\frac{\partial^2 S_2}{\partial p_2^2} = 0 \end{cases}$$

Where

$$\alpha = -\frac{p_1 - p_2}{x_1 - x_2} - 2(x_1 - x_2)$$

$$\beta = \frac{p_1 - p_2}{x_1 - x_2} - 2(x_1 - x_2)$$

In the case of a uniform distribution i.e. when $f(\theta, r) = \frac{1}{\pi}$ she obtains such an equilibrium $p_1^* = p_2^* = \frac{\pi^2}{3}$ $x_1^* = \frac{z}{2} - \frac{\pi}{4}$ $x_2^* = \frac{z}{2} + \frac{\pi}{4}$

2.2 Modification of the cost function and expanding the area of location

I consider the problem addressed in Shchiptsova [2012] with two modifications:
1. I modify the consumer cost function.
2. I consider more general case of possible locations of competitor firms: assume that there is only one street (chord) where two firms are allowed to locate, while the factory is located somewhere else in the circle at point $B(x_3, y_3)$.

We can take coordinate system such that Ox axis would be parallel to the street on which competitor firms are located i.e. the equation of the line of the street will be $y = y_1$. Assume that a consumer first goes to the factory to work and then she goes to buy the product from one of the two competitor firms. Assume that this product is heavy or bulky and therefore for a consumer to move with the product it is twice more costly than moving without this product i.e. the cost function for the consumer located at point (x, y) is:

$$C_i = (x-x_3)^2+(y-y_3)^2+(x_i-x_3)^2+(y_1-y_3)^2+p_i+2((x_i-x)^2+(y_1-y)^2) \quad i = 1, 2$$

Proposition 1 The market shares of firms do not depend on Y-coordinates of their locations.

Proof Let us find the equation of the locus (which is a line in this case) on which consumers are indifferent between two competitor firms i.e. solve

$$C_1 = C_2$$

$$(x-x_3)^2+(y-y_3)^2+(x_1-x_3)^2+(y_1-y_3)^2+p_1+2((x_1-x)^2+(y_1-y)^2) = (x-x_3)^2+(y-y_3)^2+(x_2-x_3)^2+(y_1-y_3)^2+p_2+2((x_2-x)^2+(y_1-y)^2)$$

$$p_1 + (x_1 - x_3)^2 + 2(x - x_1)^2 = p_2 + (x_2 - x_3)^2 + 2(x - x_2)^2$$

$$\bar{x} = \frac{p_2 - p_1}{4(x_2 - x_1)} + \frac{3}{4}x_1 + \frac{3}{4}x_2 - \frac{1}{2}x_3$$

As we see the way market is divided does not depend on Y-coordinates. *QED*

An interesting observation here is that the shares S_1 and S_2 in which the market is divided do not depend on y's even when three firms are not located on the same line i.e. the equilibrium shares will be the same if the street on which competitor firms are located would have been nearer to the center remaining parallel to Ox axis. Now we do the derivation which were

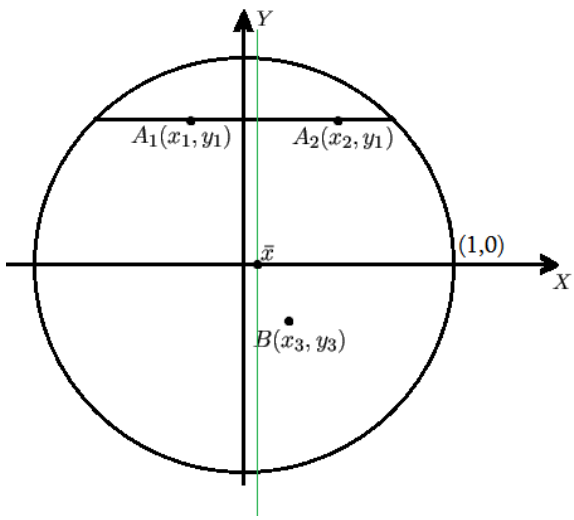

Figure 2: introducing the street

omitted by Shchiptsova in her paper and having obtained equilibrium prices and locations for our case we will see how they change.

Proposition 2: In the case when travel cost with product is higher than the travel cost without product, in Nash equilibrium competitor firms locate farther and set higher prices than in the case when travel cost is not affected by picking up the product.

Proof:

$$\frac{\partial\bar{x}}{\partial x_1} = -\frac{p_1-p_2}{4(x_1-x_2)^2} + \frac{3}{4} = (\frac{p_1-p_2}{x_1-x_2} - 3(x_1-x_2))\frac{\partial\bar{x}}{\partial p_2}$$

$$\frac{\partial\bar{x}}{\partial x_2} = \frac{p_1-p_2}{4(x_1-x_2)^2} + \frac{3}{4} = -(\frac{p_1-p_2}{x_1-x_2} + 3(x_1-x_2))\frac{\partial\bar{x}}{\partial p_2}$$

We can introduce notation $\alpha = -\frac{p_1-p_2}{x_1-x_2} - 3(x_1-x_2)$ and $\beta = \frac{p_1-p_2}{x_1-x_2} - 3(x_1-x_2)$. Then we will have $\frac{\partial S_2}{\partial x_1} = \frac{\partial S_2}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x_1} = \frac{\partial S_2}{\partial\bar{x}}\beta\frac{\partial\bar{x}}{\partial p_2} = \beta\frac{\partial S_2}{\partial p_2}$ analogically $\frac{\partial S_2}{\partial x_1} = \alpha\frac{\partial S_2}{\partial p_2}$ After putting these in (3) we will get $\frac{\partial p_2}{\partial x_1} = -\beta$ and $\frac{\partial p_1}{\partial x_2} = \alpha$

Let us calculate $\frac{\partial p_2}{\partial x_1}$ and $\frac{\partial p_1}{\partial x_2}$ using system (2). Let us introduce notation $G_1 = 1 - S_2 - p_1\frac{\partial S_2}{\partial p_1}$ and $G_2 = S_2 + p_2\frac{\partial S_2}{\partial p_2}$ then differentiating system (2) with respect to x_2 we will obtain: $\begin{cases} \frac{\partial G_1}{\partial x_2} + \frac{\partial G_1}{\partial p_1}\frac{\partial p_1}{\partial x_2} + \frac{\partial G_1}{\partial p_2}\frac{\partial p_2}{\partial x_2} = 0 \\ \frac{\partial G_2}{\partial x_2} + \frac{\partial G_2}{\partial p_1}\frac{\partial p_1}{\partial x_2} + \frac{\partial G_2}{\partial p_2}\frac{\partial p_2}{\partial x_2} = 0 \end{cases}$

So we have obtained the linear system with respect to $\frac{\partial p_2}{\partial x_1}$ and $\frac{\partial p_2}{\partial x_2}$. Using Cramer's rule we will get: $\frac{\partial p_1}{\partial x_2} = \frac{-\frac{\partial G_2}{\partial p_2}\frac{\partial G_1}{\partial x_2} + \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial x_2}}{\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial p_2} - \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial p_1}}$. Analogically differentiating system (2) with respect to x_1 and solving obtained system with respect to $\frac{\partial p_2}{\partial x_1}$ we will get: $\frac{\partial p_2}{\partial x_1} = \frac{-\frac{\partial G_2}{\partial x_1}\frac{\partial G_1}{\partial p_1} + \frac{\partial G_1}{\partial x_1}\frac{\partial G_2}{\partial p_1}}{\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial p_2} - \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial p_1}}$. Now we should substitute expressions for partial derivative of functions G_1 and G_2 which are (taking into account that $\frac{\partial S_2}{\partial p_2} = -\frac{\partial S_2}{\partial p_1}$)

$$\frac{\partial G_1}{\partial p_1} = 2\frac{\partial S_2}{\partial p_2} - p_1\frac{\partial^2 S_2}{\partial p_2^2}, \frac{\partial G_1}{\partial p_2} = -\frac{\partial S_2}{\partial p_2} + p_1\frac{\partial^2 S_2}{\partial p_2^2}$$

$$\frac{\partial G_2}{\partial p_1} = -\frac{\partial S_2}{\partial p_2} - p_2\frac{\partial^2 S_2}{\partial p_2^2}, \frac{\partial G_2}{\partial p_2} = 2\frac{\partial S_2}{\partial p_2} - p_2\frac{\partial^2 S_2}{\partial p_2^2}$$

$$\frac{\partial G_1}{\partial x_1} = -\frac{\partial S_2}{\partial x_1} + p_1\frac{\partial^2 S_2}{\partial p_2\partial x_1}, \frac{\partial G_1}{\partial x_2} = -\frac{\partial S_2}{\partial x_2} + p_1\frac{\partial^2 S_2}{\partial p_2\partial x_2}$$

$$\frac{\partial G_2}{\partial x_1} = \frac{\partial S_2}{\partial x_1} + p_2\frac{\partial^2 S_2}{\partial p_2\partial x_1}, \frac{\partial G_2}{\partial x_2} = \frac{\partial S_2}{\partial x_2} + p_2\frac{\partial^2 S_2}{\partial p_2\partial x_2}.$$

After substituting we will obtain:

$$\frac{\partial p_2}{\partial x_1} = \frac{\frac{\partial S_2}{\partial x_2}\frac{\partial S_2}{\partial p_2} + (p_1+p_2)\frac{\partial S_2}{\partial x_2}\frac{\partial^2 S_2}{\partial p_2^2} - (2p_1+p_2)\frac{\partial S_2}{\partial p_2}\frac{\partial^2 S_2}{\partial p_2\partial x_2}}{3(\frac{\partial S_2}{\partial p_2})^2 + (p_2-p_1)\frac{\partial S_2}{\partial p_2}\frac{\partial^2 S_2}{\partial p_2^2}}$$

$$\frac{\partial p_1}{\partial x_2} = \frac{-\frac{\partial S_2}{\partial x_1}\frac{\partial S_2}{\partial p_2} + (p_1+p_2)\frac{\partial S_2}{\partial x_1}\frac{\partial^2 S_2}{\partial p_2^2} - (2p_2+p_1)\frac{\partial S_2}{\partial p_2}\frac{\partial^2 S_2}{\partial p_2\partial x_2}}{3(\frac{\partial S_2}{\partial p_2})^2 + (p_2-p_1)\frac{\partial S_2}{\partial p_2}\frac{\partial^2 S_2}{\partial p_2^2}}$$

Now we should use that as we have obtained $\frac{\partial p_2}{\partial x_1} = -\beta$; $\frac{\partial p_1}{\partial x_2} = \alpha$; $\frac{\partial S_2}{\partial x_1} = \alpha\frac{\partial S_2}{\partial p_2}$; $\frac{\partial S_2}{\partial x_1} = \beta\frac{\partial S_2}{\partial p_2}$. So we will get the system:

$$\begin{cases} 2\beta\frac{\partial S_2}{\partial p_2} + 2p_2\beta\frac{\partial^2 S_2}{\partial p_2^2} - (2p_2 + p_1)\frac{\partial^2 S_2}{\partial p_2 \partial x_1} = 0 \\ 2\alpha\frac{\partial S_2}{\partial p_2} - 2p_1\alpha\frac{\partial^2 S_2}{\partial p_2^2} + (2p_1 + p_2)\frac{\partial^2 S_2}{\partial p_2 \partial x_2} = 0 \end{cases} \tag{4}$$

Let us consider the case when consumers are distributed in the city uniformly then the market share of the second firm will be $S_2 = \frac{\arccos \bar{x} - \bar{x}\sqrt{1-\bar{x}^2}}{\pi}$. So $\frac{\partial S_2}{\partial p_2} = \frac{1}{\pi}(\frac{1}{4(x_1-x_2)\sqrt{1-\bar{x}^2}} + \frac{1}{4(x_1-x_2)}\sqrt{1-\bar{x}^2} - \bar{x}\frac{1}{4\sqrt{1-\bar{x}^2}}\frac{\bar{x}}{(x_1-x_2)}) = \frac{\sqrt{1-\bar{x}^2}}{2\pi(x_1-x_2)}$ and $\frac{\partial^2 S_2}{\partial p_2^2} = \frac{\bar{x}}{8\pi\sqrt{1-\bar{x}^2}(x_1-x_2)^2}$ also $\frac{\partial^2 S_2}{\partial p_2 \partial x_2} = \frac{1}{2\pi}(\frac{1}{(x_1-x_2)^2}\sqrt{1-\bar{x}^2} - \bar{x}\frac{1}{\sqrt{1-\bar{x}^2}(x_1-x_2)}\frac{\partial \bar{x}}{\partial x_2}) = \frac{1}{x_1-x_2}\frac{\partial S_2}{\partial p_2} + \alpha\frac{\partial^2 S_2}{\partial p_2^2}$ and $\frac{\partial^2 S_2}{\partial p_2 \partial x_1} = -\frac{1}{x_1-x_2}\frac{\partial S_2}{\partial p_2} + \beta\frac{\partial^2 S_2}{\partial p_2^2}$.
So from (4) we get the system obtained by Shchiptsova:

$$\begin{cases} (2\alpha + \frac{2p_1+p_2}{x_1-x_2})\frac{\partial S_2}{\partial p_2} + p_2\alpha\frac{\partial^2 S_2}{\partial p_2^2} = 0 \\ (2\beta + \frac{2p_2+p_1}{x_1-x_2})\frac{\partial S_2}{\partial p_2} - p_1\beta\frac{\partial^2 S_2}{\partial p_2^2} = 0 \end{cases} \tag{5}$$

As the game is symmetric let us look for solutions among equal prices $p_1 = p_2$. Then we will have $\alpha = \beta$ and from the system (5) we get that $\frac{\partial^2 S_2}{\partial p_2^2} = 0$ $=> \bar{x} = 0$; $S_1 = S_2 = \frac{1}{2}$ and $-6(x_1 - x_2) + \frac{3p_1}{x_1-x_2} = 0 => p_1 = p_2 = 2(x_1 - x_2)^2$ putting this in (5) will give $x_1 - x_2 = -\frac{\pi}{2}$ on the other hand as $\bar{x} = 0 => x_1 + x_2 = \frac{2}{3}x_3$ so we will have $x_1 = \frac{1}{3}x_3 - \frac{\pi}{4}$; $x_2 = \frac{1}{3}x_3 + \frac{\pi}{4}$ and $p_1 = p_2 = \frac{\pi^2}{2}$. As we see when the travel cost with the product is higher than the travel cost without the product, the distance between firms in equilibrium is $\frac{\pi}{2}$ which is greater than the distance obtained by Shchiptsova $\frac{3\pi}{8}$, the prices have increased as well - from $\frac{3\pi^2}{16}$ to $\frac{\pi^2}{2}.QED$
So when the transportation cost of product is high firms prefer to relax the price competition by moving far from each other, set higher price of the product and as a result their profits increase.

3 Game on a square

3.1 Pure strategy equilibrium

Let us consider the city that has the form of a unit square instead of a disk. Let the street on which firms compete be parallel to one of the sides of the square. This case is almost identical to the linear - the only difference is that in the square city we don't restrict all three points to be located on the same line. Furthermore, let distribution of consumers be uniform

$$f(x,y) = \begin{cases} 1 \text{ if } x \in [0,1] \text{ and } y \in [0,1] \\ 0 \text{ otherwise} \end{cases}$$

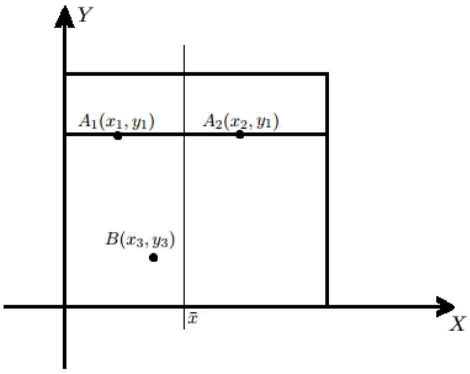

Figure 3: Location Game on a Square

It is obvious that in this case also we will have

$$\bar{x} = \frac{p_2 - p_1}{4(x_2 - x_1)} + \frac{3}{4}x_1 + \frac{3}{4}x_2 - \frac{1}{2}x_3$$

Assuming $x_1 \leq x_2$ the market shares of firms will be $S_1 = \bar{x} * 1 = \bar{x} \Rightarrow S_2 = 1 - \bar{x}$

$$\begin{cases} \Pi_1 = (\frac{p_1 - p_2}{4(x_1 - x_2)} + \frac{3}{4}x_1 + \frac{3}{4}x_2 - \frac{1}{2}x_3)p_1 \\ \Pi_2 = (1 - \frac{p_1 - p_2}{4(x_1 - x_2)} - \frac{3}{4}x_1 - \frac{3}{4}x_2 + \frac{1}{2}x_3)p_2 \end{cases}$$

We can express equilibrium prices as functions of x_1 and x_2 solving maximization problem:

$$\begin{cases} \frac{d\Pi_1}{dp_1} = \frac{2p_1 - p_2}{4(x_1 - x_2)} + \frac{3}{4}x_1 + \frac{3}{4}x_2 - \frac{1}{2}x_3 = 0 \\ \frac{d\Pi_2}{dp_2} = 1 - \frac{p_1 - 2p_2}{4(x_1 - x_2)} - \frac{3}{4}x_1 - \frac{3}{4}x_2 + \frac{1}{2}x_3 = 0 \end{cases}$$

$p_1^* = -\frac{4}{3}(x_1 - x_2)(\frac{3}{4}x_1 + \frac{3}{4}x_2 - \frac{1}{2}x_3 + 1)$
$p_2^* = -\frac{4}{3}(x_1 - x_2)(2 - \frac{3}{4}x_1 - \frac{3}{4}x_2 + \frac{1}{2}x_3)$
So we have found equilibrium prices for fixed x_1 and x_2. Now having found functions of equilibrium prices, we can try to find equilibrium locations x_1 and x_2.
$p_1 - p_2 = -\frac{4}{3}(x_1 - x_2)(\frac{3}{2}x_1 + \frac{3}{2}x_2 - x_3 - 1)$
$S_2 = 1 + \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{3} - \frac{3}{4}x_1 - \frac{3}{4}x_2 + \frac{1}{2}x_3 = \frac{2}{3} - \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{6}x_3$

$\Pi_1 = -4(x_1 - x_2)(\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{6}x_3 + \frac{1}{3})^2$ $\Pi_2 = -4(x_1 - x_2)(\frac{2}{3} - \frac{1}{4}x_1 - \frac{1}{4}x_2 - \frac{1}{6}x_3)^2$. $\Rightarrow \frac{\partial \Pi_1}{\partial x_1} = -\frac{1}{4}(x_1 + x_2 + 1)(3x_1 - x_2 + 1)$ and $\frac{\partial \Pi_2}{\partial x_2} = \frac{1}{4}(3 - x_1 - x_2)(3 + x_1 - 3x_2)$ Solving maximization problem

$$\begin{cases} \frac{\partial \Pi_1}{\partial x_1} = 0 \\ \frac{\partial \Pi_2}{\partial x_2} = 0 \end{cases}$$

we will get $x_1 = \frac{2x_3 - 1}{6}$ and$x_2 = \frac{2x_3 + 5}{6}$ As we see $x_2 - x_1 = 1$ and the length of the side of the square is also 1 so only possible location is $x_1 = 0$ and

$x_2 = 1$ and from this we get that x_3 should equal $\frac{1}{2}$ to have a symmetric equilibrium. If $x_3 = \frac{1}{2}$ The equilibrium prices will be $p_1^* = p_2^* = 2$ and consequently profits in equilibrium will be $\Pi_1 = \Pi_2 = 1$.

3.2 Mixed strategy equilibrium

Looking for a pure strategy equilibrium we have made a standard assumption that $x_1 \leq x_2$ which is made also in the original Hotelling model. On the other hand, if we do not make this assumption, the firms would know that in the equilibrium they should locate at different endpoints of the street and they will have a coordination problem as they do **not** know at which of the endpoints each of them should locate; as a result they might end up at the same point i.e. get zero profits. A solution of the coordination problem is the *mixed strategy equilibria* found by Bester et al. [1996] for the linear Hotelling model with quadratic cost function. I prove that the mixed strategy equilibrium found by Bester for origian Hotelling model with quadratic cost function holds in my model as well. Note that when $x_2 \leq x_1$ then $S_1 = 1 - \bar{x}$ and $S_2 = \bar{x}$, taking $x_3 = \frac{1}{2}$ we will have $p_1^* = -(x_2 - x_1)(x_1 + x_2 + 1)$ and $p_2^* = -(x_2 - x_1)(3 - x_1 - x_2)$.

Proposition 3: There exists a mixed strategy equilibrium when one firm mixes between locating at $(0, y_1)$ and $(1, y_1)$ - endpoints of the street with probabilities $\frac{1}{2}$ and $\frac{1}{2}$ and the other one locates at the midpoint of the street.

Proof: Without loss of generality assume that the first firm locates at $(\frac{1}{2}, y_1)$ the midpoint of the street and the second mixes between $(0, y_1)$ and $(1, y_1)$ with probabilities $\frac{1}{2}$ and $\frac{1}{2}$. Firstly, we should show that first firm's best response is to locate at the midpoint of the street given the mixed strategy of the second firm. The payoff of the first firm when the second firm is playing mixed strategy is:

$$\begin{aligned}\psi(x_1) &= \frac{1}{2}\Pi_1(p_1^*(x_1, 0), S_1(x_1, 0)) + \frac{1}{2}\Pi_1(p_1^*(x_2, 0), S_1(x_2, 0)) \\ &= \frac{1}{8}(x_1 + 2)^2(1 - x_1) + \frac{1}{8}x_1(3 - x_1)^2 \\ &= \frac{9}{8}x_1 - \frac{9}{8}x_1^2 + \frac{1}{2}\end{aligned}$$

Maximizing $\psi(x_1)$ we get $x_1 = \frac{1}{2} \Rightarrow$ the best response of the first firm, when the second firm is mixing as mentioned above, is locating at $x_1^* = (\frac{1}{2}, y_1)$. When the first firm is located at $(\frac{1}{2}, y_1)$ best responses of the second firm is to locate at $(0, y_1)$ or $(1, y_1)$ as $\frac{\partial \Pi_2}{\partial x_2} < 0$ when $x_2 \leq x_1$ and $\frac{\partial \Pi_2}{\partial x_2} > 0$ when $x_1 \leq x_2$ and $\Pi_2(\frac{1}{2}, 0) = \Pi_2(\frac{1}{2}, 1) \Rightarrow$ the second firm's best response to the first firm's strategy will be to mix between the endpoints of the street with probabilities $\frac{1}{2}$ and $\frac{1}{2}$. QED

4 Linear city - revising the *principle of maximum differentiation*

4.1 Home-job-firm-home

Now consider [0,1] linear segment city with uniformly distributed consumers. Assume that the factory where all consumers are employed is located at the point t and two competitor firms(shops) are located at x_1 and x_2 ($x_2 > x_1$). And consider the consumer cost function analogical to one defined in d'Aspremont et al. [1979]:

$$C_i = p_i + d^2(home, factory) + d^2(factory, firm_i) + d^2(firm_i, home)$$

Let us calculate the location $\bar{x}$ of the indifferent consumer i.e. solve
$p_1 + (\bar{x} - t)^2 + (x_1 - t)^2 + (x_1 - \bar{x})^2 = p_2 + (\bar{x} - t)^2 + (x_2 - t)^2 + (x_2 - \bar{x})^2$
from this we obtain

$$\bar{x} = \frac{p_1 - p_2}{2(x_1 - x_2)} + x_1 + x_2 - t$$

Now we should solve profit maximization problem for both firms:

$$\begin{cases} \frac{\partial \Pi_1}{\partial p_1} = 0 \\ \frac{\partial \Pi_2}{\partial p_2} = 0 \end{cases}$$

Solving this system we obtain $p_1^* = \frac{-2}{3}(x_1 - x_2)(x_1 + x_2 + t)$ and $p_2^* = \frac{2}{3}(x_1 - x_2)(x_1 + x_2 - t - 2)$. $\Rightarrow \Pi_1 = \bar{x}p_1^* = \frac{-2}{9}(x_1 - x_2)(x_1 + x_2 + t)^2$ and $\Pi_2 = (1 - \bar{x})p_2^* = \frac{-2}{9}(x_1 - x_2)(x_1 + x_2 - 2 - t)^2$. Solving profit maximization problem with respect to locations for both firms

$$\begin{cases} \frac{\partial \Pi_1}{\partial x_1} = 0 \\ \frac{\partial \Pi_2}{\partial x_2} = 0 \end{cases}$$

is equivalent to

$$\begin{cases} \frac{-2}{9}(x_1 + x_2 + \frac{1}{2})(3x_1 - x_2 + t) = 0 \\ \frac{-2}{9}(x_1 + x_2 - \frac{5}{2})(x_1 - 3x_2 + 2 + t) = 0 \end{cases}$$

From this system (checking second order conditions) we obtain the equilibrium locations: $x_1^* = \frac{1-t}{4}$ and $x_2^* = \frac{3+t}{4}$.
So if we take the symmetric case when $t = \frac{1}{2}$ we will obtain $x_1^* = \frac{1}{8}$ and $x_2^* = \frac{7}{8}$. This means that the principle of maximum differentiation does not hold in this case.

4.2 Home-job-home-firm-home

Now let us assume that after job consumers first go home and after that go to firm(shop). It is obvious that this case is identical to one considered in d'Aspremont et al.(1979). Lets find indifferent consumer's location for this case:

$$p_1+(\hat{x}-t)^2+(\hat{x}-t)^2+(\hat{x}-x_1)^2+(\hat{x}-x_1)^2 = p_2+(\hat{x}-t)^2+(\hat{x}-t)^2+(\hat{x}-x_2)^2+(\hat{x}-x_2)^2$$

$$\hat{x} = \frac{p_1 - p_2}{4(x_1 - x_2)} + \frac{x_1 + x_2}{2}$$

Running the same procedure we will obtain $p_1^* = \frac{-2(x_1-x_2)(x_1+x_2+2)}{3}$ and $p_2^* = \frac{2(x_1-x_2)(x_1+x_2-4)}{3}$. So the profits of the firms will be $\pi_1 = \frac{-1}{9}(x_1 - x_2)(x_1+x_2+2)^2$ and $\pi_2 = \frac{1}{9}(x_1-x_2)(4-x_1-x_2)^2$. For this case $\frac{\partial \pi_1}{\partial x_1} < 0$ and $\frac{\partial \pi_2}{\partial x_2} > 0$ i.e. firms will locate at boundary points of the city (D'Aspremont et. al (1979) result) this result is known as *the principle of maximum differentiation.* But what does this mean? If we allow locating outside the city then following this principle equilibrium should be $-\infty$ and $+\infty$ but they are $x_1^* = -\frac{1}{4}$ and $x_2^* = \frac{5}{4}$. I.e. in equilibrium firms choose to locate outside the market which cannot be explained by any economic intuition so the famous classical result of d'Aspremont et al. is inconsistent with economic intuition.

4.3 Two types of consumers

Assume that with probability s a consumer after job goes directly to a firm. And with probability $(1 - s)$ consumer after job first goes home and then goes to a firm. After firm everybody goes home. Like Economides [1986] we are interested when locations in equilibrium "are strictly interior to [0, 1]"

Proposition 4:In Nash equilibrium firms will locate inside the city (not at extreme points) if $s > \frac{1}{4t}$.

Proof:The expected location of indifferent consumer will be

$$\tilde{x} = s\bar{x} + (1-s)\hat{x} = \frac{s+1}{2}\left(\frac{p_1 - p_2}{2(x_1 - x_2)} + x_1 + x_2\right) - ts$$

Solving profit maximization problem with respect to prices gives

$$p_1^* = \frac{2(x_1 - x_2)(2ts - 2 - (x_1 + x_2)(s+1))}{3(s+1)}$$

and

$$p_2^* = \frac{-2(x_1 - x_2)(4 + 2ts - (x_1 + x_2)(s+1))}{3(s+1)}$$

Using p_1^* and p_2^* in profit maximization problem with respect to locations we obtain (checking second order conditions) the equilibrium locations $x_1^* = \frac{4ts-1}{4(s+1)}$ and $x_2^* = \frac{5+4ts}{4(s+1)}$. Solving $x_1^* > 0$ gives $s > \frac{1}{4t}$. If we take $t = \frac{1}{2}$ we get that firms will locate inside the city (not on boundary points) iff the share (probability) of consumers going after job directly to firm is more than one half.

5 Conclusion

In this paper I have discussed how Hotelling's model of a linear city could be developed into a bit more realistic and complex models by introducing the points of attraction and modifying the transportation cost. I have generalized model of Shchiptsova [2012] by allowing firms to locate at any chord of the disk city. As a result I have shown that if the transportation of the product is costlier than just travel cost then competitors locate farther from each other and set higher prices. It was also important to see that the famous *principle of maximum differentiation*, in contrast to Economides [1986], does not hold even in the models of linear cities with quadratic transportation cost functions, and therefore, it is valid only for very specific setting of duopolistic competition. All these help to better understand the agglomeration process, namely, the fact that neither principle of minimum nor maximum differentiation could be used for explaining locations of competitor firms in the real world.

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