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TAIL RISKS, ASSET PRICES, AND INVESTMENT HORIZONS

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Tail Risks, Asset Prices, and Investment Horizons

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Abstract:
We examine how extreme market risks are priced in the cross-section of asset returns at various horizons. Based on the frequency decomposition of covariance between indicator functions, we define the quantile cross-spectral beta of an asset capturing tail-specific as well as horizon-, or frequency-specific risks. Further, we work with two notions of frequency-specific extreme market risks. First, we define tail market risk that captures dependence between extremely low market as well as asset returns. Second, extreme market volatility risk is characterized by dependence between extremely high increments of market volatility and extremely low asset return. Empirical findings based on the datasets with long enough history, 30 Fama-French Industry portfolios, and 25 Fama-French portfolios sorted on size and book-to-market support our intuition. Results suggest that both frequency-specific tail market risk and extreme volatility risks are significantly priced and our five-factor model provides improvement over specifications considered by previous literature.

JEL: C21, C58, G12
Keywords: Asset pricing, downside risk, frequency-specific risk, tail risk

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1 Introduction

Classical result of asset pricing literature states that price of an asset should be equal to its expected discounted payoff. In the Capital Asset Pricing Model (CAPM) introduced by Sharpe (1964), Lintner (1965), Black (1972), we assume that stochastic discount factor can be approximated by return on market portfolio and thus expected excess returns can be fully described by their market betas based on covariance between asset return and market return. Yet, decades of the consequent research show that we are unable to sufficiently explain the cross-section of asset returns with this notion. Instead, literature calls for more accurate characterization of risks associated with assets that will better reflect preferences of investors. We aim to show that in order to understand formation of expected returns, one has to look into some special features of asset returns that are crucial in terms of preferences of a representative investor. We argue that the two important features are risk related to tail events, and frequency-specific risk. To characterize the risks, we derive novel quantile cross-spectral representation of beta. Our work nests classical representation that simply averages beta with equal weights over different quantile levels, as well as frequencies.

Economists has long recognized that decisions under risk are more sensitive to changes in probability of possible extreme events compared to probability of a typical event. The expected utility might not reflect this behavior since it weights probability of outcomes linearly. Consequently, CAPM beta as an aggregate measure of risk may fail to explain the cross-section of asset returns. Several alternative notions emerged in the literature. Mao (1970) presents survey evidence showing that decision makers tend to think of risk in terms of the possibility of outcomes below some target. For an expected utility maximizing investor, Bawa and Lindenberg (1977) has provided a theoretical rationale for using lower partial moment as a measure of portfolio risk. Based on the rank-dependent expected utility due to Yaari (1987), Polkovnichenko and Zhao (2013) introduce utility with probability weights and derive corresponding pricing kernel. More recently, Ang et al. (2006); Lettau et al. (2014) argue that downside risk – risk of negative returns – is priced across asset classes and is not captured by CAPM betas. Further, Farago and Tédongap (2017) extend the results using general equilibrium model based on generalized disappointment aversion and shows that downside risks in terms of market return and market volatility are priced in the cross-section of asset returns.[1]

The results described above leads us to question appropriateness of the expected utility maximizers in asset pricing. A recent strand of literature solves the problem by considering quantile of the utility instead of expectation. This literature complements the literature focusing on downside risk as it highlights the notion of economic agents particularly averse to outcomes below some threshold compared to outcomes above this threshold. The concept of a quantile maximizer and its features was proposed by Manski (1988), and later axiomatized by Rostek (2010). Most recently, de Castro and Galvao (2017) develop a model of quantile optimizer in a dynamic setting. A different approach to emphasizing investor's aversion towards least favorable outcomes defines theory based on Choquet expactations. This approach is based

[1] In addition, it is interesting to note that equity and variance risk premium are also associated with compensation for jump tail risk (Bollerslev and Todorov, 2011). More general exploration of asymmetry of stock returns is provided by Ghysels et al. (2016), who propose a quantile-based measure of conditional asymmetry and show that stock returns from emerging markets are positively skewed. Conrad et al. (2013) use option price data and find a relation between stock returns and their skewness. Another notable approach uses high frequency data to define realized semivariance as a measure of downside risk (Barndorff-Nielsen et al., 2008). From a risk-measure standpoint, dealing with negative events, especially rare events, is highly discussed theme in both practice and academics. The most prominent example is Value-at-Risk (Adrian and Brunnermeier, 2016; Engle and Manganelli, 2004).

on distortion function that alters probability distribution of future outcomes by accentuating probabilities associated with least desirable outcomes. This approach was utilized in finance, for example, by Bassett Jr et al. (2004).

Whereas aggregating linearly weighted outcomes may not reflect the sensitivity of investors to tail risk, aggregating linearly weighted outcomes over various frequencies, or economic cycles may not reflect risk specific to different investment horizons. One can suspect that an investor cares differently about short-term and long-term risk according to their preferred investment horizon. Distinguishing between long-term and short-term dependence between economic variables was proven to be an insightful approach since the introduction of co-integration (Engle and Granger, 1987). Frequency decomposition of risk thus uncovers another important feature of risk which cannot be captured solely by market beta which captures risk averaged over all frequencies. This recent approach to asset pricing enables to shed light on asset returns and investor's behaviour from a different point of view highlighting heterogeneous preferences. Empirical justification is brought by Boons and Tamoni (2015) and Bandi and Tamoni (2017) who show that exposure in long-term returns to innovations in macroeconomic growth and volatility of matching half-life is significantly priced in variety of asset classes. The results are based on decomposition of time series into components of different persistence proposed by Ortu et al. (2013). Piccotti (2016) further sets portfolio optimization problem into frequency domain using matching of utility frequency structure and portfolio frequency structure, and Chaudhuri and Lo (2016) present approach to constructing mean-variance-frequency optimal portfolio. This optimization yields mean-variance optimal portfolio for a given frequency band, and thus optimizes portfolio for a given investment horizon.

From a theoretical point of view, preferences derived by Epstein and Zin (1989) enables to study frequency aspects of investor's preferences, and quickly became a standard in the asset pricing literature. With the important results of Bansal and Yaron (2004), long-run risk started to enter asset pricing discussions. Dew-Becker and Giglio (2016) investigate frequency-specific prices of risk for various models and conclude that cycles longer than business cycle are significantly priced in the market. Other papers utilizes frequency domain and Fourier transform to facilitate estimation procedures for parameters hard to estimate using conventional approaches. Berkowitz (2001) generalizes band spectrum regression and enables to estimate dynamic rational expectations models matching data only in particular ways, for example, forcing estimated residuals to be close to white noise. Dew-Becker (2016) proposes spectral density estimator of long-run standard deviation of consumption growth, which is a key component for determining risk premiums under Epstein-Zin preferences, and shows its superior performance compared to the previous approaches. Crouzet et al. (2017) develop model of multi-frequency trade set in frequency domain and show that restricting trading frequencies reduces price informativeness at those frequencies, reduces liquidity and increases return volatility.

The debate clearly indicates that the standard assumptions leading to classical asset pricing models do not correspond with reality. In this paper, we suggest that more general pricing models have to be defined and they should take into consideration both asymmetry of dependence structure among stock market, and different behavior of investors at various investment horizons.

The main contribution of this paper is threefold. First, we define a simple theoretical model in which the representative investor cares differently about long- and short-term risk associated with an asset. Moreover, we propose an extension of the model that incorporates notion of aversion to losses into the investor's decision making. This model then leads to the five-factor representation of the risk premium and is used for building our empirical model.

Second, based on the frequency decomposition of covariance between indicator functions, we define the quantile cross-spectral beta of an asset capturing tail-specific as well as frequency-specific risks. The newly defined notion of beta can be viewed as disaggregation of a classical beta to a frequency-, and tail- specific beta. With this notion, we examine how extreme market risks are priced in the cross-section of asset returns at various horizons. We define frequency-specific tail market risk that captures dependence between extremely low market and asset returns, as well as extreme market volatility risk that is characterized by dependence between extremely high increments of market volatility and extremely low asset return.

Third, based on the quantile cross-spectral betas, we define five-factor model that provides considerable improvements in explaining cross-section of asset returns. Results on a 30 Fama-French Industry portfolios, and 25 Fama-French portfolios sorted on size and book-to-market suggest extreme market risk is priced in cross section of asset returns and it is differently priced for long and short horizon. This extreme market risk is characterized by the risk of extremely low returns or extremely high volatility.

The rest of the paper has the following structure. Section 2 defines a simple theoretical model of the representative investor's preferences. Section 3 introduces concept of quantiles cross-spectral betas later employed in defining empirical models. Section 5 defines the empirical models used for testing significance of extreme risks. Section 6 conducts the empirical analysis of the extreme risks and provides definition of tested robustness checks. Section 7 concludes. In Appendix we report some robustness checks and give details on estimating quantile cross-spectral betas.

2 Economic model

Our goal is to show that extreme risk is priced in cross-section of asset returns. Specifically, we focus on two types of extreme risk: tail market risk, and extreme volatility risk. Further, we are interested to decompose the tail risks into frequencies to be able to define short- and long-run extreme risks. We start the discussion with theoretical motivation using simple economic model and show how the aversion of tail market risk which varies with different investment horizons may emerge. In the next section, we propose a method how to robustly measure these kind of risks.

2.1 General framework

The aim of this subsection is to show how the frequency dependent utility may emerge. We present a general notion of a consumption (wealth) process that consists of two parts: short-, and long-term. In our setting, the corresponding utility function is additive in those two parts of consumption. This feature then leads to the stochastic discount factor which is influenced by both parts of the consumption.

Let's assume that some function of the consumption process is sum of two parts, which are functions of short- and long-term components of the consumption

$$f(c_t) = f_S(c_{t,S}) + f_L(c_{t,L}) \tag{1}$$

where c_t is consumption at time t and $c_{t,S}$ and $c_{t,L}$ are short- and long-term parts of the consumption[2]. Of course, the simplest case is when the consumption is additive in both these

[2]This general framework nests, for example, long-run risk model proposed by Bansal and Yaron (2004). It can be simply shown that the log of consumption is sum of log of the short-term part and long-term part.

parts

$$c_t = c_{t,S} + c_{t,SL}, \tag{2}$$

and these two parts are orthogonal to each other

$$\mathbb{C}ov_t(c_{t+1,S}, c_{t+1,L}) = 0. \tag{3}$$

For the simplicity reasons, we will use this simple case throughout the rest of the paper. Based on this decomposition, the representative agent cares differently about the short term and long term part[3]. The utility from the consumption is function of its short-term part and long-term part and it is additive in utility from these parts

$$u(c_t) = u(c_{t,S}, c_{t,L}) = u_S(c_{t,S}) + u_L(c_{t,L}). \tag{4}$$

In the conventional setting, the representative agent maximizes expected value of infinite sum of discounted utilities subject to the budget constraints in the form of

$$\begin{aligned} U_t &= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}), \\ s.t. \quad c_t &= e_t - p_t \xi_t \\ c_{t+1+j} &= e_{t+j} + d_{t+j} \xi_t \end{aligned} \tag{5}$$

where $\{e_{t+j}\}$ is an endowment process, $\{d_{t+j}\}$ is a stream of payoffs which is purchased by the investor at time t for price p_t in quantity ξ_t. On the other hand, in our approach, we assume two parts of consumption process and the utility in additive form of these two parts, so the long-term objective changes to

$$\begin{aligned} U_t &= \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \big(u_S(c_{t+j,S}) + u_L(c_{t+j,L}) \big) \\ s.t. \quad c_{t+j} &= c_{t+j,S} + c_{t+j,L} \\ c_t &= \alpha_t(e_t - p_t\xi_t) + (1-\alpha_t)(e_t - p_t\xi_t) \\ c_{t+1+j} &= \alpha_{t+i}(e_{t+i} + d_{t+i}\xi_t) + (1-\alpha_{t+i})(e_{t+i} + d_{t+i}\xi_t) \end{aligned} \tag{6}$$

where α_t determines proportion of the consumption due to the short-term fluctuations ($c_{t,S}$) and the rest of the consumption, given by $1-\alpha_t$, is due to the long term trend in the consumption ($c_{t,L}$).

By obtaining the first order condition with respect to ξ_t, we may solve the problem for price of the asset

$$p_t = \mathbb{E}_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j}. \tag{7}$$

where we have used the fact that the partial derivative of the utility at time t with respect to ξ_t is

$$\begin{aligned} \frac{\partial u(c_t)}{\partial \xi_t} &= u'_S\big(\alpha_t(e_t - p_t\xi_t)\big)(-p_t\alpha_t) + u'_L\big((1-\alpha_t)(e_t - p_t\xi_t)\big)(-p_t(1-\alpha_t)) \\ &= -p_t\big((\alpha_t)u'_S(c_{t,S}) + (1-\alpha_t)u'_L(c_{t,L})\big) \\ &= -p_t u'(c_t). \end{aligned} \tag{8}$$

[3] This setting is very similar to the one of Bandi et al. (2018).

Analogously, we can obtain derivatives of c_{t+1+j} in expectations.

Knowing that 7 holds at time $t+1$

$$p_{t+1} = \mathbb{E}_{t+1} \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+1+j})}{u'(c_{t+1})} d_{t+1+j}, \tag{9}$$

and using law of iterated expectations, we can rewrite the price at time t as

$$\begin{aligned} p_t &= \mathbb{E}_t\Big[\beta \frac{u'(c_{t+1})}{u'(c_t)} d_{t+1} + \beta \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+1+j})}{u'(c_t)} d_{t+1+j}\Big] & (10)\\ &= \mathbb{E}_t\Big[\beta \frac{u'(c_{t+1})}{u'(c_t)} d_{t+1} + \beta \frac{u'(c_{t+1})}{u'(c_t)} \sum_{j=1}^{\infty} \beta^j \mathbb{E}_{t+1}\big[\frac{u'(c_{t+1+j})}{u'(c_{t+1})} d_{t+1+j}\big]\Big] & (11)\\ &= \mathbb{E}_t\Big[\beta \frac{u'(c_{t+1})}{u'(c_t)} (d_{t+1} + p_{t+1})\Big]. & (12) \end{aligned}$$

This can be rewritten using well-known stochastic discount factor (SDF) and payoff at time $t+1$ as

$$p_t = \mathbb{E}_t[m_{t+1} x_{t+1}] \tag{13}$$

where $x_{t+1} \equiv d_{t+1} + p_{t+1}$ and the SDF is defined as $m_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}$.

We can observe that the SDF can be split into sum of two parts, each part depending on only one part of the consumption process

$$\begin{aligned} m_{t+1} &= m_{t+1,S}(c_{t+1,S}) + m_{t+1,L}(c_{t+1,L}) & (14)\\ &= \beta \frac{\alpha_{t+1} u'_S(c_{t+1,S})}{(\alpha_t) u'_S(c_{t,S}) + (1-\alpha_t) u'_L(c_{t,L})} + \beta \frac{(1-\alpha_{t+1}) u'_S(c_{t+1,L})}{(\alpha_t) u'_S(c_{t,S}) + (1-\alpha_t) u'_L(c_{t,L})} & (15) \end{aligned}$$

By dividing both sides of 13 by p_t, defining the gross return as $R_{t+1} \equiv \frac{x_{t+1}}{p_t}$, using the fact that $\mathbb{E}[XY] = \mathbb{C}ov[XY] + \mathbb{E}[X]\mathbb{E}[Y]$, and the fact that risk-free rate is given by $R^f = \frac{1}{\mathbb{E}_t[m_{t+1}]}$, we can rewrite it as

$$\mathbb{E}_t[R_{t+1}] - R_t^f = -R_t^f \mathbb{C}ov_t(m_{t+1}, R_{t+1}). \tag{16}$$

Using 14 and knowing that $\mathbb{C}ov[X+Y, Z] = \mathbb{C}ov[X, Z] + \mathbb{C}ov[Y, Z]$, we can split the last equation into

$$\mathbb{E}_t[R_{t+1}] - R_t^f = -R_t^f \Big(\mathbb{C}ov_t(m_{t+1,S}, R_{t+1}) + \mathbb{C}ov_t(m_{t+1,L}, R_{t+1})\Big). \tag{17}$$

Moreover, following Bandi et al. (2018), we assume that the return process of the asset also consists of two parts which are orthogonal to each other

$$\begin{aligned} R_t &= R_{t,S} + R_{t,L}, & (18)\\ \mathbb{C}ov_t(R_{t+1,S}, R_{t+1,L}) &= 0 & (19) \end{aligned}$$

and that the short-term and long-term parts of the consumption process and return process are orthogonal to each other, too, which leads to the following

$$\begin{aligned} \mathbb{C}ov_t(R_{t+1,S}, c_{t+1,L}) &= 0 & (20)\\ \mathbb{C}ov_t(R_{t+1,L}, c_{t+1,S}) &= 0. & (21) \end{aligned}$$

The risk premium then can be rewritten as

$$\mathbb{E}_t[R_{t+1}] - R_t^f = -R_t^f \Big(\mathbb{C}ov_t(m_{t+1,S}, R_{t+1,S}) + \mathbb{C}ov_t(m_{t+1,L}, R_{t+1,L}) \Big) \tag{22}$$

We can see that the risk premium of an asset is given by two covariances. One is given by the covariance between short-term part of the return process and a function of the short-term consumption part. Second is defined analogously using long-term parts. Thus, the risk premium is given by short-term dependence between consumption process and return process, and long-term dependence.

2.2 Specific example

We present here a specific candidate for the utility function which is dependent on both parts of the consumption process separately. It leads to the model in which the risk premium is given by weighted linear combination of beta for the short-term part of the consumption process, and beta for the long-term part.

2.2.1 Power utility

We argue that the possible utility function of the representative agent is sum of two power utility functions, i.e.

$$u(c_t) = u(c_{t,S}, c_{t,L}) = u_S(c_{t,S}) + u_L(c_{t,L}) = \frac{c_{t,S}^{1-\gamma_S}}{1-\gamma_S} + \frac{c_{t,L}^{1-\gamma_L}}{1-\gamma_L}. \tag{23}$$

This separation enables to model different risk attitude towards the long-term part and short-term part of the consumption. One can be a long-term investor and be more risk averse to the long-term risk than to the short-term fluctuations and this framework enables to model this explicitly. We now derive the results which were discussed in the previous subsection for some general utility function $u(c_t)$.

The partial derivative of the utility at time t with respect to ξ_t is

$$\frac{\partial u(c_t)}{\partial \xi_t} = (\alpha_t(e_t - p_t\xi_t))^{-\gamma_S}(-p_t\alpha_t) + ((1-\alpha_t)(e_t - p_t\xi_t))^{\gamma_L}(-p_t(1-\alpha_t)) \tag{24}$$

$$= -p_t(\alpha_t)(c_t^S)^{-\gamma_S} + (1-\alpha_t)(c_t^L)^{-\gamma_L}) \tag{25}$$

$$= -p_t u'(c_t), \tag{26}$$

and the same applies to the differentiation of $u(c_{t+1+j})$ in expectations.

By substituting $u'(c_{t+1}) = \alpha_{t+1}c_{t+1,S}^{-\gamma_S} + (1-\alpha_{t+1})c_{t+1,L}^{-\gamma_L}$ into 22 and after few algebraic operations, we obtain the following

$$\begin{aligned}\mathbb{E}_t[R_{t+1}] - R_t^f = &\frac{\mathbb{C}ov_t(\alpha_{t+1}c_{t+1,S}^{-\gamma_S}, R_{t+1,S})}{\mathbb{V}ar_t(\alpha_{t+1}c_{t+1,S}^{-\gamma_S})}\Big(- \frac{\mathbb{V}ar_t(\alpha_{t+1}c_{t+1,S}^{-\gamma_S})}{\mathbb{E}_t(u'(c_{t+1}))}\Big) + \\ &\frac{\mathbb{C}ov_t((1-\alpha_{t+1})c_{t+1,L}^{-\gamma_L}, R_{t+1,L})}{\mathbb{V}ar_t((1-\alpha_{t+1})c_{t+1,L}^{-\gamma_S})}\Big(- \frac{\mathbb{V}ar_t((1-\alpha_{t+1})c_{t+1,L}^{-\gamma_S})}{\mathbb{E}_t(u'(c_{t+1}))}\Big)\end{aligned} \tag{27}$$

which in terms of betas reduces to

$$\mathbb{E}_t[R_{t+1}^i] - R_t^f = \beta_{i,S,t}\lambda_{S,t} + \beta_{i,L,t}\lambda_{L,t} \tag{28}$$

where we stress that the beta coefficient (quantity of risk) is specific for every asset i, and lambda (price of risk) is common for every asset. We can see that $\beta_{i,S,t}$ is a function of covariance between transformed short-term part of the consumption and short-term part of asset return, and $\beta_{i,L,t}$ is a function of covariance between the transformed long-term part of the consumption and long-term part of asset return.

To summarize it, we expect that price of risk of short-term and long-term part will differ. As stated in Bandi et al. (2018), frequency is a dimension of risk and thus investors may care differently about various parts of it.

2.3 Implementing tail risk aversion

There are many ways how to introduce loss or tail-risk aversion into the economic model defined in 6. We will propose two approaches how this can be done. First one is based on the asymmetric utility function and the second one on the utility functional over the whole infinite stream of the consumption process which overweights the left-tail events.

2.3.1 Asymmetric utility function

One of the possibilities how to introduce the aversion to losses into our economic model is to define some utility function over consumption which incorporates this feature. We will do that by adapting the utility function based on the one from the cumulative prospect theory of Tversky and Kahneman (1979) and Tversky and Kahneman (1992) which posses a kink at some reference point[4]. The original utility function of Tversky and Kahneman (1979) is in the form of

$$v(x) = x^{\alpha} I\{x \geq 0\} - \lambda(-x)^{\beta} I\{x < 0\} \tag{29}$$

where the utility is not computed directly over the consumption (wealth) but over gain or loss $x = c - c_0$ relative to a reference point c_0; so, in terms of consumption this can be rewritten as

$$v(c) = (c - c_0)^{\alpha} I\{c \geq c_0\} - \lambda(-(c - c_0))^{\beta} I\{c < c_0\}. \tag{30}$$

In our setting, we may assume that the reference point is given by the value of some quantile of distribution of consumption, i.e. $c_0 = q_c(\tau)$ where q_c is quantile function of the consumption.

Because of the presence of the kink at the reference point, the original utility function is not differentiable at the point of the kink. This feature makes the utility hard to incorporate into the intertemporal setting. So, instead of using the original utility function of Tversky and Kahneman (1979), we use the specification introduced in Hung and Wang (2005)

$$v(c) = (1 - e^{-\beta(c-c_0)}) I\{c \geq c_0\} - \lambda(1 - e^{\frac{\beta}{\lambda}(c-c_0)}) I\{c < c_0\}. \tag{31}$$

In our setting, the utility of the economic agent defined in 4 is computed as a sum of two utilities; these utilities are computed over different parts of the consumption - short- and long-term. This means that both parts of the utility posses the form of 31 with distinct values of

[4] In the setting of the original prospect theory Tversky and Kahneman (1979) and cumulative prospect theory Tversky and Kahneman (1992), not only the utility function posses a kink in the reference point, but also the probabilities are non-linearly transformed mimicking the loss aversion. We do not aim to fully incorporate this theory into our model as it is not of the main interest of this paper.

parameters β and λ

$$u_S(c_S) = (1 - e^{-\beta_S(c_S - c_{0,S})})I\{c_S \geq c_{0,S}\} - \lambda_S(1 - e^{\frac{\beta_S}{\lambda_S}(c_S - c_{0,S})})I\{c_S < c_{0,S}\} \tag{32}$$

$$u_L(c_L) = (1 - e^{-\beta_L(c_L - c_{0,L})})I\{c_L \geq c_{0,L}\} - \lambda_L(1 - e^{\frac{\beta_L}{\lambda_L}(c_L - c_{0,L})})I\{c_L < c_{0,L}\} \tag{33}$$

and the overall utility is sum of those two utilities

$$u(c) = u_S(c_S) + u_L(c_L) \tag{34}$$

where the reference values are computed using the same value of τ for computing reference point as a $\tau-$quantile of the respective part of the consumption. Parameters β and λ may be the same for both parts of the consumption, or they may be different to introduce different degrees of loss aversion for each part of the consumption.

Derivative of 31 with respect to the corresponding part of the consumption is computed as

$$u'_h(c_h) = \beta e^{-\beta(c_h - c_{0,h})}I\{c_h \geq c_{0,h}\} + \beta e^{\frac{\beta}{\lambda}(c_h - c_{0,h})}I\{c_0 < c_{0,h}\},\ h = \{S, L\} \tag{35}$$

where we can see that the utility posses a derivative at every point. Knowing that 8 holds, we can substitute this into 22 to obtain excess return representation

$$\begin{aligned}\mathbb{E}_t[R_{t+1}] - R^f = \frac{\mathbb{C}ov_t(\alpha_{t+1}u'_S(c_{t+1,S}), R_{t+1,S})}{\mathbb{V}ar_t(\alpha_{t+1}u'_S(c_{t+1,S}))}\Big(- \frac{\mathbb{V}ar_t(\alpha_{t+1}u'_S(c_{t+1,S}))}{\mathbb{E}_t(u'(c_{t+1}))}\Big) + \\ \frac{\mathbb{C}ov_t((1-\alpha_{t+1})u'_L(c_{t+1,L}), R_{t+1,L})}{\mathbb{V}ar_t((1-\alpha_{t+1})u'_L(c_{t+1,L}))}\Big(- \frac{\mathbb{V}ar_t((1-\alpha_{t+1})u'_S(c_{t+1,S}))}{\mathbb{E}_t(u'(c_{t+1}))}\Big)\end{aligned} \tag{36}$$

yielding the same beta representation as in the case of power utility model but additionally the consumption is weighted differently for values below quantile threshold and above it making the agent averse to the losses or event left tail events.

Moreover, using the formula for the derivative of each part of the utility, we can decompose the last equation into

$$\begin{aligned}&\mathbb{E}_t[R_{t+1}] - R^f = \\ &\frac{\mathbb{C}ov_t\big(\alpha_{t+1}\beta_S e^{-\beta_S(c_S - c_{0,S})}I\{c_S \geq c_{0,S}\}, R_{t+1,S}\big)}{\mathbb{V}ar_t\big(\alpha_{t+1}\beta_S e^{-\beta_S(c_S - c_{0,S})}I\{c_S \geq c_{0,S}\}\big)}\Big(- \frac{\mathbb{V}ar_t\big(\alpha_{t+1}\beta_S e^{-\beta_S(c_S - c_{0,S})}I\{c_S \geq c_{0,S}\}\big)}{\mathbb{E}_t(u'(c_{t+1}))}\Big) + \\ &\frac{\mathbb{C}ov_t\big((1-\alpha_{t+1})\beta_L e^{-\beta_L(c_L - c_{0,L})}I\{c_L \geq c_{0,L}\}, R_{t+1,L}\big)}{\mathbb{V}ar_t\big((1-\alpha_{t+1})\beta_L e^{-\beta_L(c_L - c_{0,L})}I\{c_L \geq c_{0,L}\}\big)}\Big(- \frac{\mathbb{V}ar_t\big((1-\alpha_{t+1})\beta_L e^{-\beta_L(c_L - c_{0,L})}I\{c_L \geq c_{0,L}\}\big)}{\mathbb{E}_t(u'(c_{t+1}))}\Big) + \\ &\frac{\mathbb{C}ov_t\big(\alpha_{t+1}\beta_S e^{\frac{\beta_S}{\lambda_S}(c_S - c_{0,S})}I\{c_S < c_{0,S}\}, R_{t+1,S}\big)}{\mathbb{V}ar_t\big(\alpha_{t+1}\beta_S e^{\frac{\beta_S}{\lambda_S}(c_S - c_{0,S})}I\{c_S < c_{0,S}\}\big)}\Big(- \frac{\mathbb{V}ar_t\big(\alpha_{t+1}\beta_S e^{\frac{\beta_S}{\lambda_S}(c_S - c_{0,S})}I\{c_S < c_{0,S}\}\big)}{\mathbb{E}_t(u'(c_{t+1}))}\Big) + \\ &\frac{\mathbb{C}ov_t\big((1-\alpha_{t+1})\beta_L e^{\frac{\beta_L}{\lambda_L}(c_L - c_{0,L})}I\{c_L < c_{0,L}\}, R_{t+1,L}\big)}{\mathbb{V}ar_t\big((1-\alpha_{t+1})\beta_L e^{\frac{\beta_L}{\lambda_L}(c_L - c_{0,L})}I\{c_L < c_{0,L}\}\big)}\Big(- \frac{\mathbb{V}ar_t\big((1-\alpha_{t+1})\beta_L e^{\frac{\beta_L}{\lambda_L}(c_L - c_{0,L})}I\{c_L < c_{0,L}\}\big)}{\mathbb{E}_t(u'(c_{t+1}))}\Big).\end{aligned} \tag{37}$$

That can be rewritten in terms of betas and lambdas as

$$\mathbb{E}_t[R_{t+1}] - R^f_t = \beta^U_{i,S,t}\lambda^U_{S,t} + \beta^U_{i,L,t}\lambda^U_{L,t} + \beta^L_{i,S,t}\lambda^L_{S,t} + \beta^L_{i,L,t}\lambda^L_{L,t} \tag{38}$$

where the superscript indicates the part of the distribution of the consumption where we measure the dependence (U for upper part of the distribution given by the values above the kink, and L for lower part).

2.3.2 Rank-dependent intertemporal utility functional

Another way how to incorporate aversion to losses into the economic model is to transform the probabilities associated with the continuation value of the utility regardless the contemporaneous utility function employed. We will briefly introduce the model based on Chew and Epstein (1990), which is slightly modified for our purposes. In their specification, based on the nonexpected utility preferences, they work with utility functional of Yaari (1987) defined on $D(R)$, the set of cumulative distribution functions (c.d.f.'s) on the real line, in the form

$$V_Y(F) = \int zd(g(F(z))) = \int zg'(F(z))dF(z), \quad F \in D(R) \tag{39}$$

where g is a probability weighting function such that $g : [0, 1] \to [0, 1]$ is continuous and strictly increasing, $g(0) = 0$, and $g(1) = 1$. By denoting probability weighting density as $W \equiv g'$, we can rewrite the utility functional as

$$V_Y(F) = \mathbb{E}[zW(F(z))], \tag{40}$$

which belongs to the class of rank-dependent expected utility (RDEU). Following Chew and Epstein (1990), we employ the utility functional 39 to the model 6 and obtain the following

$$U_{Y,t} = V_Y(F_{\Sigma_t}) \tag{41}$$

where we define $\Sigma_t \equiv \sum_{j=0}^{\infty} \beta^j \big(u_S(c_{t+j,S}) + u_L(c_{t+j,L})\big)$ and F_{Σ_t} is its c.d.f. We can rewrite it in the form

$$U_{Y,t} = \mathbb{E}_t\big[\Sigma_t W(F(\Sigma_t))\big]. \tag{42}$$

If the $g(F(z)) = F(z)$, then 42 reduces to the standard specification of 6 based on the expected utility. On the other hand, W may be defined in the way that it strongly overweights the left tail outcomes making the agent extremely averse to the left-tail outcomes. This specification clearly depends on both parts of the consumption process and the utility is influenced by the weighting feature of the rank dependent utility functional.

3 Measuring quantile-frequency risk

In this section, we propose methods how to robustly measure the relation between asset return and some economic variable over some specific horizon in a given part of the joint distribution. The aim is not to precisely estimate the theoretical models proposed earlier, but to introduce general measures that estimate risk in various parts of the joint distribution and over various investment cycles. Frequency part is important because of the fact that the risk premium in the setting described above is determined by the covariance between asset return and two parts of the consumption process - short- and long-term. The measure of dependence between asset return and consumption in specific part of the distribution is important because agent is highly averse to extremely low outcomes and thus requires a premium for assets that posses high covariance with weighted consumption.

First, we define quantile risk measure based on covariance between indicator functions, which has natural economic interpretation in terms of probabilities. Second, we introduce frequency decomposition, and combine these two frameworks into quantile cross-spectral risk measure, which are the building blocks for our empirical model. These measure enable us to robustly test for the presence of extreme market risks over various horizons in the asset prices. The aim is not to convince the reader that the functional form of the preferences follows precisely our model, but to show that there is a heterogeneity in the weights that investors put to the risk for different investment horizons and different parts of the distribution of their future wealth. By estimating prices of risk for short- and long-term part, we are able to identify the horizon the investor care most about, and by estimating prices of risk for various parts of the distribution, we are able to identify the part of the joint distribution towards which is the investor the most risk averse. This is done by controlling for CAPM beta and the influence of these new measures is measured as an incremental information over simplifying assumptions that lead to the CAPM beta asset pricing models.

3.1 Tail risk

Based on many simplifying assumptions, asset pricing theory states that risk premium of an asset or portfolio can be explained by its covariance with some reference economic or financial variable such as consumption growth or return on market portfolio. This measure may not be sufficient if the assumptions are wrong, mainly in the cases in which the the investor cares about different parts of the distribution of his future wealth differently. For example, as discussed in Ang et al. (2006), if the investor's decisions are characterized by the rational disappointment utility function, classical covariance-based measure of dependence cannot fully explain asset prices. Hence, the most widely used measure of dependence between two variables $r_{t,i}$ and $r_{t,j}$, cross-covariance,

$$\gamma_k^{r_i,r_j} = \mathbb{C}ov\big(r_{t+k,i}, r_{t,j}\big) \equiv \mathbb{E}[(r_{t+k,i} - \bar{r}_i)(r_{t,j} - \bar{r}_j)], \tag{43}$$

is due to its averaging nature unable to describe asymmetry features of dependence structure between two variables unless the variables are jointly normal. If we want to measure dependence separately in different parts of a distribution - and obtain dependence measure in various parts of joint distribution, we have to employ more flexible measures. Since we are interested in pricing extreme negative events, we want to measure dependence and risk in lower quantiles of the joint distribution. We propose to use quantity of the following form

$$\gamma_k^{r_i,r_j}(\tau_{r_i}, \tau_{r_j}) \equiv \mathbb{C}ov\big(I\{r_{t+k,i} \leq q_{r_i}(\tau_{r_i})\}, I\{r_{t,j} \leq q_{r_j}(\tau_{r_j})\}\big), \tag{44}$$

where $r_{t,i}$ and $r_{t,j}$ are two time series of strictly stationary random variables, $q_X(\tau)$ is a quantile function of random variable X, $\tau_i, \tau_j \in (0, 1)$, and $I\{A\}$ is indicator function of event A. The measure is given by the covariance between two indicator functions and can fully describe joint distribution of the pair of random variables r_i and r_j. If distribution functions of r_i and r_j are continuous, the quantity is essentially difference between copula of pair r_i and r_j and independent copula, thus the following quantity $Pr\big(r_{t+k,i} \leq q_{r_i}(\tau_{r_i}), r_{t,m} \leq q_{r_m}(\tau_{r_m})\big) - \tau_{r_i}\tau_{r_i m}$. Thus, covariance between indicators measures additional information from the copula over independent copula about how likely is that the series are jointly less or equal to their given quantiles. It enables to flexibly measure both cross-sectional structure and time-series structure of the pair of random variables.

The quantity introduced in Eq. 44 can be further generalized in the way that one can replace quantiles of respective variables by some general threshold values

$$\gamma_k^{r_i,r_j}(Q_{r_i}, Q_{r_j}) \equiv \mathbb{C}ov\big(I\{r_{t+k,i} \leq Q_{r_i})\}, I\{r_{t,j} \leq Q_{r_j}\}\big) \tag{45}$$

where Q_{r_i} and Q_{r_j} are general threshold values, which do not necessary need to be equal. These threshold values may be derived from distribution of some reference variable. Since we are interested in explaining risk premiums of assets, we follow the usual setting and denote returns of some asset or portfolio i as $r_{i,t}$, and returns of market portfolio denoted as $r_{m,t}$.

In our model, we set threshold values to be equal, $Q_{r_i} = Q_{r_m}$ and are derived from distribution of market returns. These values are given by τ_{r_m} unconditional quantile of market returns

$$Q_{r_m} = q_{r_m}(\tau) \tag{46}$$

and thus our measure of dependence between asset i and market return can be written as

$$\gamma_k^{r_i,r_m}(\tau) \equiv \mathbb{C}ov\big(I\{r_{t+k,i} \leq q_{r_m}(\tau))\}, I\{r_{t,m} \leq q_{r_m}(\tau)\}\big). \tag{47}$$

Simple tail risk beta (not decomposed into horizons) is defined using measure given in 47 for $k = 0$ and normalized by variance of the indicator function of the market return

$$\beta^{r_i,r_m}(\tau) \equiv \frac{\gamma_0^{r_i,r_m}}{\gamma^{r_m}} = \frac{\mathbb{C}ov\big(I\{r_{t,i} \leq q_{r_m}(\tau))\}, I\{r_{t,m} \leq q_{r_m}(\tau)\}\big)}{\mathbb{V}ar(I\{\{r_{t,m} \leq q_{r_m}(\tau)\}\})} \tag{48}$$

$$= \frac{\mathbb{C}ov\big(I\{r_{t,i} \leq q_{r_m}(\tau))\}, I\{r_{t,m} \leq q_{r_m}(\tau)\}\big)}{\tau(1-\tau)} \tag{49}$$

This definition of beta will be used in the simple model defined later.

3.2 Frequency-specific risk

It is natural to assume that economic agents care differently about long-, and short-term investment horizon in terms of expected returns and associated risk. Investors may be interested in long-term profitability of their portfolio and do not concern with short-term fluctuations. One possibility how the preferences may look like is in the Subsection 2.1. Frequency-dependent features of an asset return then play an important role for an investor. Bandi and Tamoni (2017) argues that covariance between two returns can be decomposed into covariance between disaggregated components evolving over different time scales, and thus the risk on these components can vary. Hence, market beta can be decomposed into linear combination of betas measuring dependence at various scales, i.e.dependence between fluctuations with various half-lives. Frequency specific risk at given time plays an important role for determination of asset prices, and the price of risk in various frequency bands may differ, i.e. the expected return can be decomposed into linear combination of risks in various frequency bands.

The most simple and natural way how to decompose covariance between two assets into dependencies over different horizons is via its Fourier and inverse Fourier transform. Frequency domain counterpart of cross-covariance is obtained as Fourier transform of the cross-covariance functions. Conversely, cross-covariance can be obtained from inverse Fourier transform of its cross-spectrum in the following way

$$S^{r_i,r_m}(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \gamma_k^{r_i,r_m} e^{-ik\omega}$$

$$\gamma_k^{r_i,r_m} = \int_{-\pi}^{\pi} S^{r_i,r_m}(\omega) e^{ik\omega} d\omega$$

where $S^{r_i,r_m}(\omega)$ is cross-spectral density of random variables r_i and r_m, $\gamma_k^{r_i,r_m}$ is cross-covariance function given by equation 43. It is important to note that cross-covariance can be decomposed into frequencies, more specifically, for $k = 0$, we can decompose covariance between two time series into the covariance components at each frequency ω

$$\mathbb{C}ov(r_i, r_m) = \int_{-\pi}^{\pi} S^{r_i,r_m}(\omega)d\omega.$$

Following the same logic decomposition of variance follows as

$$Var(r_i) = \int_{-\pi}^{\pi} S^{r_i}(\omega)d\omega.$$

where $S^{r_i}(\omega)$ is spectrum of r_i.

Since we can decompose cross-covariance between two returns into covariances at each frequency, we can disentangle the dependence at short- and long-term components. Then, beta for an asset i and factor m can be decomposed to as

$$\beta^{r_i,r_m} \equiv \frac{Cov(r_i, r_m)}{Var(r_m)} = \int_{-\pi}^{\pi} w(\omega)\frac{S^{r_i,r_m}(\omega)}{S^{r_m}(\omega)}d\omega = \int_{-\pi}^{\pi} w(\omega)\beta^{i,m}(\omega)d\omega \tag{50}$$

where $w(\omega) = \frac{S^{r_m}(\omega)}{Var(r_m)}$ represent weights. The decomposition is important step since it provides decomposition of classical beta into the weighted frequency-specific betas. Using similar approach, Bandi and Tamoni (2017) estimate price of risk for different investment horizons and show that investors posses heterogeneous preferences over various economic cycles instead of looking only on averaged quantities over the whole frequency spectrum.

4 Quantile-frequency specific risk

Since we argue that both tail risk as well as frequency-specific risk are important in explaining formation of asset returns, we aim to combine these risks into a single model. We start by defining measure of risk associated with various combinations of quantile and frequency in order to determine the most important combination priced across assets.

Our measures of risk in the quantile-frequency domain are based on the dependence measures recently introduced by Baruník and Kley (2015). To quantify risk premium across frequencies and across the joint distribution, we use the quantile cross-spectral densities to build a quantile cross-spectral beta. Both these points are explained in more detail in Section 5.

4.1 Quantile cross-spectral beta

The cornerstone of the new beta representation lies in quantile cross-spectral density kernels which are defined as

$$f^{r_i,r_m}(\omega; \tau_{r_i}, \tau_{r_m}) \equiv \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \gamma_k^{r_i,r_m}(\tau_{r_i}, \tau_{r_m})e^{-ik\omega} \tag{51}$$

where $\tau_{r_i}, \tau_{r_m} \in [0, 1]$. A quantile cross-spectral density kernel is obtained as a Fourier transform of covariances of indicator functions defined in Equation 44, and will allow us to define beta that will capture the tail risks as well as frequency specific risks. As stated earlier, in our model we will use as a threshold value the τ unconditional quantile of market returns and thus we will work with

$$f^{r_i,r_m}(\omega; \tau) \equiv \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \gamma_k^{r_i,r_m}(\tau)e^{-ik\omega}. \tag{52}$$

A quantile cross-spectral (QS) betas in the general case in which the threshold values are given by τ_{r_i} quantile of asset returns and τ_{r_m} quantile of market returns are defined as

$$\beta^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m}) \equiv \frac{f^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m})}{f^{r_m}(\omega,\tau_{r_m})} \equiv \frac{\sum_{k=-\infty}^{\infty}\gamma_k^{r_i,r_m}(\tau_{r_i},\tau_{r_m})e^{-ik\omega}}{\sum_{k=-\infty}^{\infty}\gamma_k^{r_m}(\tau_{r_m})e^{-ik\omega}}. \tag{53}$$

QS betas for given asset quantify the dependence between asset i and market factor m for a given frequency ω at chosen quantiles τ_{r_i} and τ_{r_m} of the joint distribution. Once again, we will work with the following quantities

$$\beta^{r_i,r_m}(\omega;\tau) \equiv \frac{f^{r_i,r_m}(\omega;\tau)}{f^{r_m}(\omega,\tau)} \equiv \frac{\sum_{k=-\infty}^{\infty}\gamma_k^{r_i,r_m}(\tau)e^{-ik\omega}}{\sum_{k=-\infty}^{\infty}\gamma_k^{r_m}(\tau)e^{-ik\omega}}. \tag{54}$$

We can also construct beta for a given frequency band, accordingly

$$\beta^{r_i,r_m}(\Omega;\tau) \equiv \int_{\Omega} \frac{f^{r_i,r_m}(\omega;\tau)}{f^{r_m}(\omega,\tau)} d\omega \tag{55}$$

where $\Omega \equiv [\omega_1,\omega_2)$, $\omega_1,\omega_2 \in [-\pi,\pi]$, $\omega_1 < \omega_2$ is a frequency band. This definition is important since it allows to define short-run, or long-run bands covering corresponding frequencies, and hence disaggregate beta based on the specific demands of a researcher.

4.2 Quantile cross-spectral beta under Gaussianity

Before we continue and use the new beta representation, it is important to note how newly defined quantity relates to a classical beta under the assumption of Gaussian distribution, as commonly assumed by many asset pricing models. Assuming that returns of an asset and returns of market portfolio are jointly normal random variables independently distributed through the time (correlated Gaussian white noises), QS betas would be in the following form

$$\beta_{Gauss}^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m}) = \frac{C_{Gauss}(\tau_{r_i},\tau_{r_m};\rho) - \tau_{r_i}\tau_{r_m}}{\tau_{r_m}(1-\tau_{r_m})} \tag{56}$$

where C_{Gauss} is Gaussian copula with correlation coefficient ρ. This stems from the fact that quantile cross-spectral density corresponds to a difference of probabilities $Pr\big(r_{t,i} \leq q_{r_i}(\tau_{r_i}), r_{t,m} \leq q_{r_m}(\tau_{r_m})\big) - \tau_{r_i}\tau_{r_im}$, where $\{\tau_{r_i},\tau_{r_im}\}$ are probability levels under Gaussian distribution.

In our case, the threshold values are given by the market quantile. So, if we want to compute beta for some asset i and market under the Gaussian distribution assumption, the the value of τ_{r_i} has to be estimated, first. We do that using the empirical distribution function $\hat{F}_{r_i}$ of asset i's returns, i.e. $\tau_{r_i} = \hat{F}_{r_i}(q_{r_m}(\tau))$. The rest is the same as in the case described earlier.

QS betas are constant over frequencies under Gaussian white noise assumption, and depend only on chosen quantiles and correlation coefficient between asset and market return. Hence Eq. 56 provides the quantile cross-spectral counterpart to classical CAPM beta as these are equivalent. We will use this fact to construct our model later. In the spirit of Ang et al. (2006) and Lettau et al. (2014), we define relative QS betas which capture additional information not contained in the classical CAPM beta.

Finally, we note that for serially uncorrelated variables (no matter of their joint or marginal distributions), the Frećhet/Hoeffding bounds gives the limits that QS beta can attain

$$\frac{\max\{\tau_{r_i}+\tau_{r_m}-1,0\} - \tau_{r_i}\tau_{r_m}}{\tau_{r_m}(1-\tau_{r_m})} \leq \beta^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m}) \leq \frac{\min\{\tau_{r_i},\tau_{r_m}\} - \tau_{r_i}\tau_{r_m}}{\tau_{r_m}(1-\tau_{r_m})}.$$

5 Pricing extreme risks across frequency domain

Quantile cross-spectral betas defined in the previous section will be the cornerstone of the empirical model defined in this section. Using the theoretical motivation, we assume that QS betas for low values of threshold values will be significant determinants of risk. We expect that weighting density strongly overweights the outcomes from the left part of the distribution of future outcomes. Using QS betas, we define pricing model encompassing tail market risk and extreme volatility risk. Both these risks are further decomposed into long- and short-term components in order to obtain their prices of risk separately.

Tail market risk (TR) represents dependence between extreme negative events of both market as well as asset return. It differs from downside risk used in Ang et al. (2006); Lettau et al. (2014) since downside betas are computed based on covariates of asset return with a market return being under some threshold value. In contrast, QS betas captures risk that both market as well as asset return will be extremely unfavorable. In other words, it captures joint probability that market as well as asset returns will be below some threshold level.

Extreme market volatility risk (EVR) captures unpleasant situations in which extremely high increments of market volatility are linked to the extremely low asset asset returns. We argue that both these risks are significant determinants of risk of an asset and thus should be priced in cross-section of asset returns.

Values of τ_{r_i}, percentage value for the quantiles for asset thresholds, are not explicitly fixed to quantile of their returns because we do not explicitly care about dependence between quantile values in the cross-section. We rather care about dependence in extreme market situations. Thus the threshold values for asset returns are given by values of quantile of market returns; these threshold values are same for all the assets, which corresponds to different quantiles for each asset. Formally, for each portfolio we obtain threshold values as a τ_{r_i} quantile of its distribution where $\tau_{r_i} = F_{r_i}(q_{r_m}(\tau_{r_m}))$. Let's consider a model in which we set threshold value to be equal to 5% quantile of market return. Value of τ_{r_m} in Equation 53 is equal to 5% but τ_{r_i} must be estimated. First, this 5% market quantile must be transformed using empirical cumulative distribution functions into probability that given asset return is below this value for each asset, and then the QS betas are computed as $\beta^{r_i,r_m}(\omega;\tau^i_{r_i},\tau_{r_m})$ where $\tau^i_{r_i}$ differs across assets (for one asset 5% quantile of market return may correspond to 1% quantile of its distribution, for another asset it may correspond to 8% quantile of its distribution). Same logic is applied to both tail market risk betas and extreme volatility risk betas. By setting market return and portfolio threshold equal, we avoid problem of potential data-mining. Potentially better fit could be obtained by finding threshold values with the best model fit for a specific dataset, but may not be robust across datasets.

Regarding the frequency decomposition of the risks, we specify our models to include disaggregation of risk into two horizons - long and short. Long horizon is defined by corresponding frequencies of cycles of 1.5 year and longer, and short horizon by frequencies of cycles shorter than 1.5 year. Procedure how to obtain these betas is explained in Section 6.

In each of the models defined in the paper we control for CAPM beta as a baseline measure of risk. This ensures that if the QS betas are significant determinants of risk premium, they do not simply duplicate information contained in CAPM beta. Moreover, in case of tail market risk, we define relative betas that explicitly capture only the additional information over CAPM beta. Throughout the paper we impose the restriction that market price of risk is correctly priced implying that it is equal its average return.

5.1 Tail market risk

We assume that dependence between market return and asset return during extreme negative events is priced across assets. The stronger the relationship between market and asset during an extreme events is, the bigger the risk premium investors demand. Tail market risk is a direct extension of downside risk discussed above. Whereas downside risk captures risk of negative events, tail risk is connected to negative events with more severe impact.

Because we want to quantify risk which is not captured by CAPM beta, we propose to test significance of tail market risk via differences of the estimated QS beta and QS beta implied by the Gaussian white noise assumption. We call this difference *relative* QS betas. For a given frequency band Ω_j and given quantiles τ_{r_i} and τ_{r_m}, the relative beta is defined as follows

$$\beta_{rel}^{r_i,r_m}(\Omega_j;\tau_{r_i},\tau_{r_m}) \equiv \beta^{r_i,r_m}(\Omega_j;\tau_{r_i},\tau_{r_m}) - \beta_{Gauss}^{r_i,r_m}(\Omega_j,\tau). \tag{57}$$

In our case, beta is function of frequency band Ω_j and threshold value given by τ quantile of market returns

$$\beta_{rel}^{r_i,r_m}(\Omega_j;\tau) \equiv \beta^{r_i,r_m}(\Omega_j;\tau) - \beta_{Gauss}^{r_i,r_m}(\Omega_j,\tau). \tag{58}$$

Relative QS betas measure additional information not captured by classical CAPM beta. In case the CAPM beta captures all information, and returns are Gaussian, the relative QS beta will be zero at all frequencies and quantiles.

Our first model is a three-factor market model which contains only tail market risk, and is defined as

$$\mathbb{E}[r_i^e] = \sum_{j=1}^{2} \beta_{rel}^{r_i,r_m}(\Omega_j;\tau)\lambda^{TR}(\Omega_j;\tau) + \beta_i^{CAPM}\lambda^{CAPM}, \tag{59}$$

where β_i is classical CAPM beta, λ^{CAPM} is price of risk for market risk captured by the classical beta, and $\lambda^{TR}(\Omega_j,\tau)$ is price of tail risk (TR) for given quantile and given frequency band. We impose restriction that market risk is correctly priced, i.e. λ^{CAPM} is equal to average market return, and portfolio threshold is the same as market threshold and, $\tau_{r_i} = F_{r_i}(q_{r_m}(\tau_{r_m}))$. If asset returns do not posses features of deviations from assumptions mentioned above, then the relative betas will be equal to zero and thus all the information about dependency during extreme events is captured by CAPM betas. On the other hand, if there is a significant difference between information captured by CAPM beta and QS betas, then the difference will be nonzero and may be priced in cross section of asset returns, which will be assessed based on significance of related prices of risk.

This model directly relates to the model proposed in the Equation 38. More specifically, we approximate the betas for the lower part of the distribution of consumption (superscript L) with QS betas and for the upper part of the distribution with CAPM beta. Although the CAPM beta captures dependence over the whole distribution, because the QS betas are estimated mostly for very low parts of the distribution, the additional information contained in the CAPM beta (in comparison with betas defined for the upper part of the distribution) can be neglected.

5.2 Extreme volatility risk

Volatility risk is important risk priced across assets. Ang et al. (2006) document that assets with high sensitivities to innovations in aggregate volatility have low average returns. Because of the fact that time of high volatility within the economy is perceived as a time with high uncertainty, investors are willing to pay more for the assets that yield high returns during these turmoils and thus positively covary with innovations in market volatility. This drives the prices of these assets up and decreases expected returns. In addition, decomposition of volatility into short-run and long-run when determining asset premium was proven to be useful as well (Adrian and Rosenberg, 2008). Moreover, Bollerslev et al. (2016) incorporated notion of downside risk into concept of volatility risk and showed that stocks with high negative realized semivariance yield higher returns. Farago and Tédongap (2017) examine downside volatility risk in their five-factor model and obtain model with negative prices of risk of volatility downside factor yielding low returns for assets that positively covary with innovations of market volatility during disappointing events.

We assume that assets that yield highly negative returns during times of large innovations of volatility are less desirable for investors and thus should be rewarded by holding these assets. For simplicity reasons, we estimate market volatility using basic GARCH(1,1) model [5] and obtain estimates of squared volatility. Then the changes in squared volatility are calculated as

$$\Delta\sigma_t^2 \equiv \sigma_t^2 - \sigma_{t-1}^2. \tag{60}$$

Because of the nature of covariance between indicator functions, we work with negative differences of the volatility, $-\Delta\sigma_t^2$, then the high volatility increments correspond to low quantiles of distribution of the negative differences. We investigate whether dependence between extreme market volatility and tail events of assets is priced across assets. Threshold values for portfolio returns are obtained in the same manner as for tail market risk and are derived from distribution of market returns, $\tau_{r_i} = F_{r_i}(q_{r_m}(\tau_{r_m}))$. For example, for model with $\tau_{r_m} = 0.05$, extreme market volatility beta is computed using threshold for innovations of market squared volatility as 5% quantile of its distribution of negative values (corresponding to 95% quantile of the original distribution), and threshold for portfolio returns is computed as 5% quantile of distribution of market returns.

Three-factor model containing extreme volatility risk betas solely is defined as

$$\mathbb{E}[r_i^e] = \sum_{j=1}^{2} \beta^{r_i,\Delta\sigma^2}(\Omega_j;\tau)\lambda^{EV}(\Omega_j;\tau) + \beta_i^{CAPM}\lambda^{CAPM}, \tag{61}$$

where we also impose restriction that market risk is correctly priced, i.e. λ^{CAPM} is equal to average market return.

5.3 Full five-factor model

Finally, we combine the risks into a single five-factor model that includes both tail market risk and extreme volatility risk for both short- and long-run horizons, as well as market risk associated with classical CAPM beta. Model posses the following form

$$\begin{aligned} \mathbb{E}[r_i^e] = \sum_{j=1}^{2} \beta_{rel}^{r_i,r_m}(\Omega_j;\tau)\lambda^{TR}(\Omega_j;\tau) + \beta_i^{CAPM}\lambda^{CAPM} \\ + \sum_{j=1}^{2} \beta^{r_i,\Delta\sigma^2}(\Omega_j;\tau)\lambda^{EV}(\Omega_j;\tau) \end{aligned} \tag{62}$$

[5] As a robustness check, we compute volatility as realized volatility from daily data

where we restrict λ^{CAPM} to be equal to the average market return. We remind that the market threshold is equal to portfolio threshold. This means that τ_{r_m} is given and τ_{r_i} is computed for each portfolio from its respective empirical distribution. Threshold value for extreme volatility risk is given by quantile of distribution of differences of market volatility, and for given model, tau for extreme volatility risk is the same as tau for tail market risk, $\tau_{\Delta\sigma^2} = \tau_{r_m}$, and portfolio threshold is the same as for tail market risk.

Throughout the paper, we focus on results for τ_{r_m} equal to 5% and 10% (models denoted as QS05 and QS10). In addition, we report various results for 1%, 15%, and 25% quantiles. Moreover, root mean squared pricing error of the fitted models is reported for continuum of quantiles between 1% and 50% for completeness. The choice of 5% and 10% quantiles is natural and arises in many economic and finance applications. Probably the most prominent example is Value-at-Risk, which a benchmark measure of risk widely used in practice and studied among academics.

5.4 Simple three-factor model

As an intermediary step, we define model which contains both tail market risk and extreme market volatility risk but does not take into consideration frequency decomposition. It posses the following form

$$\mathbb{E}[r_i^e] = \beta_{rel}^{r_i,r_m}(\tau)\lambda^{TR}(\tau) + \beta_i^{CAPM}\lambda^{CAPM} + \beta^{r_i,\Delta\sigma^2}(\tau)\lambda^{EV}(\tau) \tag{63}$$

where we define quantile betas as

$$\beta^{r_i,r_m}(\tau_{r_i}, \tau_{r_m}) \equiv \frac{\gamma_0^{r_i,r_j}}{\gamma^{r_m}} \tag{64}$$

where r_m stands either for return on market portfolio, or changes in negative of squared market volatility. Relative beta in case of TR is defined as difference between quantile beta and beta defined under normality assumption

$$\beta_{rel}^{r_i,r_m}(\tau) \equiv \beta^{r_i,r_m}(\tau) - \beta_{Gauss}^{r_i,r_m}(\omega;\tau) \tag{65}$$

where beta under normality assumption is the same as in Equation 56 since it does not depend on frequency. Threshold values are obtained in the same way as in case of Full 5-factor model.

6 Testing for quantile-frequency specific risk

In this section, we estimate the models defined in the previous section and assess whether the tail risks are priced in the cross section of asset returns and whether we capture new features of priced risk not described by other competing models.

6.1 Estimation of QS betas

Estimation of QS betas (for both TR and EVR) relies on proper estimation of quantile cross-spectral densities using rank-based copula cross-periodograms, which are then smoothed in order to obtain consistency of the estimator. Technical details are provided in the Appendix A. Betas from the simplified model defined in Equation 63 are simply estimated using empirical distribution function of the market return distribution.

6.2 Fama-MacBeth regressions

To test our models, we employ procedure of Fama and MacBeth (1973). In the first stage, we estimate all required QS betas, relative QS betas, and CAPM betas for all portfolios. We define two non-overlapping horizons: short and long. Horizon is specified by the corresponding frequency band. We specify long horizon by frequencies with corresponding cycles 1.5 year and longer, and short horizon by frequencies with corresponding cycles below 1.5 year. QS betas for these horizons are obtained by averaging QS betas over these frequency bands

$$\begin{aligned} \beta^{r_i,r_m}(\Omega_L;\tau) &\equiv \frac{1}{n_L}\sum_{i=1}^{n_L}\beta^{r_i,r_m}(\omega_i^L;\tau) \\ \beta^{r_i,r_m}(\Omega_S;\tau) &\equiv \frac{1}{n_S}\sum_{i=1}^{n_S}\beta^{r_i,r_m}(\omega_i^S;\tau) \end{aligned} \tag{66}$$

where Ω_L (Ω_S) is frequency band for long (short) horizon, and $\omega_i^L \in \Omega_L$ ($\omega_i^S \in \Omega_S$).

In the second stage, we use these betas as explanatory variables and regress average portfolio returns on them. We assess significance of a given risk by significance of the corresponding price of risk [6]. Thus, in the second stage in case of the Full 5-factor full model, we estimate model of the following form

$$\begin{aligned} \bar{r}_i^e = &\sum_{j=1}^{2}\hat{\beta}_{rel}^{r_i,r_m}(\Omega_j;\tau)\lambda^{TR}(\Omega_j;\tau) + \hat{\beta}_i^{CAPM}\lambda^{CAPM} \\ &+\sum_{j=1}^{2}\hat{\beta}^{r_i,\Delta\sigma^2}(\Omega_j;\tau)\lambda^{EV}(\Omega_j;\tau) + e_i. \end{aligned} \tag{67}$$

The same estimation logic applies to the simplified three-factor model.

As mentioned earlier, we estimate our models for various values of threshold value given by τ quantile of market return. In the scatter plots of actual and predicted returns, we focus on versions of our model where $\tau = 0.05$ (QS05) and $\tau = 0.10$ (QS10). We compare the results for our models with i) classical CAPM ii) downside risk model by Ang et al. (2006) (DR1) iii) GDA3 and GDA5 models by Farago and Tédongap (2017). Performance of all models is assessed based on their root mean squared pricing error (RMSPE), which is widely used metric for assessing model fit in asset pricing literature. All the competing models are estimated for comparison purposes without any restrictions except that the market price of risk is correctly priced (equal to the average market return over the observed period) using OLS. Thus, GDA3 and GDA5 are despite their theoretical background estimated without setting any restriction to their coefficients and are also estimated in two stages.

6.3 Data

To illustrate the main findings, we use 30 Fama-French industry portfolios data monthly sampled between July 1926 and November 2017 (1097 observations). These data satisfy the need of our model to posses long enough history in order to obtain reliable results. In Appendix D, we report also results for 25 Fama-French portfolios sorted on size and book to market over the same time span. Regarding the market data, instead of using consumption data, we follow

[6] As shown in Shanken (1992), if the betas are estimated over the whole period, the second stage regression is T-consistent.

τ_{r_m}	Tail market risk λ^{TR}_{long}	λ^{TR}_{short}	λ^{CAPM}	RMSPE	Extreme volatility risk λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	0.18	1.14	0.66	14.66	0.32	-1.01	0.66	23.61
	(3.50)	(7.32)			(7.85)	(-1.61)		
0.05	0.86	1.10	0.66	20.15	0.45	-3.26	0.66	17.70
	(4.29)	(1.82)			(11.72)	(-2.84)		
0.1	1.38	0.85	0.66	19.65	0.55	-2.61	0.66	16.44
	(5.19)	(1.14)			(14.62)	(-2.22)		
0.15	1.53	1.26	0.66	17.51	0.71	-3.64	0.66	14.41
	(8.45)	(1.58)			(14.97)	(-3.35)		
0.25	2.36	1.24	0.66	21.24	0.93	-4.06	0.66	16.64
	(6.77)	(1.14)			(14.14)	(-2.91)		

Table 1: Estimated coefficients. Prices of risk of two versions of three-factor model estimated on monthly data of 30 Fama-French equal-weighted industry portfolios sampled between July 1927 and November 2017. Models are estimated for various values of thresholds. Market price of risk is imposed to be equal to the average market return.

Campbell (1993) and use data on broad market index to avoid problems connected to the consumption data. Excess market return is computed using value-weighted average return on all CRSP stocks and Treasury bill rate from Ibbotson Associates[7].

We also estimate our models using daily data. Various models do not perform very well on a daily sampling frequency and we want to assess whether this is the case for our models, too. We employ the same datasets as in the main analysis but with daily frequency. The sampling period for daily data is between July 1926 and March 2019. The performance in this case will be also compared with above mentioned competing models.

6.4 Estimation results

6.4.1 Three-factor models

We report estimation results of the three-factor models in the Table 1. To take into account multiple hypothesis testing, we follow Harvey et al. (2016) and report t-statistics of estimated parameters (in parenthesis). Regarding the TR model, beta for short-horizon is more significant for τ_{r_m} being equal to 0.01, in the rest of the cases, beta for long horizon is more significant. In case of the EVR model, beta for the long-horizon is more significant determinant of risk premium for all the values of τ_{r_m} in comparison to short-horizon beta. We can see that the TR model model outperforms the EVR model for τ_{r_m} being equal to 0.01.

6.4.2 Full model

As a preliminary investigation, we conduct an analysis in which we examine tail risk and extreme volatility risk without taking into consideration the frequency aspect. Estimated coefficients can be found in left panel of Table 2. We can observe that tail risk is significantly priced across low quantiles with expected positive sign. Extreme volatility risk is significantly priced for 10%, 15%, and 25% quantiles suggesting that investors price dependence between assets and market volatility, but focus on more probable market situations. RMSPE of the model for various market threshold defined as τ quantile of market return is depicted in left panel of Figure 2. We can deduce that better fit is obtained for lower values of thresholds and for very low τ it

[7] All the data were obtained from Kenneth French's online data library.

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.56	0.35	0.66	15.57	-0.12	1.14	0.22	0.02	0.66	13.74
	(10.96)	(0.80)			(-0.71)	(7.01)	(1.88)	(0.06)		
0.05	2.41	0.48	0.66	20.78	-0.45	1.27	0.50	-2.93	0.66	15.87
	(4.72)	(0.55)			(-0.96)	(2.49)	(2.86)	(-2.71)		
0.1	2.07	1.61	0.66	19.69	-0.15	0.81	0.51	-2.23	0.66	15.94
	(3.48)	(2.34)			(-0.29)	(1.25)	(3.52)	(-1.82)		
0.15	3.67	1.58	0.66	19.85	0.61	0.72	0.46	-3.51	0.66	13.30
	(4.80)	(1.88)			(1.60)	(1.13)	(3.50)	(-3.35)		
0.25	4.50	3.26	0.66	25.44	0.69	0.71	0.68	-3.86	0.66	15.61
	(4.15)	(2.34)			(1.35)	(0.85)	(4.37)	(-2.83)		

Table 2: Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 30 Fama-French equal-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return.

can even outperform GDA5 model which is a 5-factor model. For higher values of τ, RMSPE of our simple model exceeds RMSPE of GDA5 model suggesting that indeed extreme risks of the assets are priced factor.

Estimated parameters of the full model can be found in the right panel of Table 2. We observe that significant determinants of the risk are short tail risk and long extreme volatility risk, both significantly priced across portfolios with expected signs. Tail risk is more significant for lower values of τ meaning that dependence between market return and portfolio return during extremely negative events is a significantly determinant of risk premium. On the other hand, long-run extreme volatility risk is significantly priced across all values of τ_{r_m}, but becomes more prominent for higher values of the quantile. We can deduce that price of long-run risk of Bansal and Yaron (2004) is hidden in this coefficient. Coefficients of the prices of risk for long tail risk and short extreme volatility risk posses negative sign, which may seem counterintuitive. This may suggests that investors are extremely averse to long-run dependence between extremely negative returns and high volatility but at the same time exposure to the extreme volatility risk in the short run is desirable as the prices will adjust to the market turmoil quickly. Tail risk in the long run for lower quantiles is also negative but the coefficients are not significant.

In Figure 1, we compare performance of our QS models, QS05 ($\tau_{r_m} = 0.05$) and QS10 ($\tau_{r_m} = 0.10$), with various other models. It is notable that CAPM, and DR1 model completely fail to price the portfolios, better fit and lower RMSPE is obtained by GDA3 and GDA5 models. Finally, the best fit is provided by our QS models since returns lie closer to the 45 degree line. Right panel of Figure 2 depicts performance of the QS model against market thresholds given by τ quantile of market distribution. We observe better performance of our model in comparison to GDA5 model for all threshold values below 30% market quantile, and generally very good performance for low values of threshold suggesting that extreme risks are significant determinants of risk premium.

6.5 Disentangling model performance

We will answer the question whether the performance of our model is driven by the quantile decomposition of the risk only, or the frequency decomposition brings a significant improvement over the simple specification. To do that, we employ horse race between betas from the Simple model and betas from the Full model, and assess the significance of the prices of risk for given

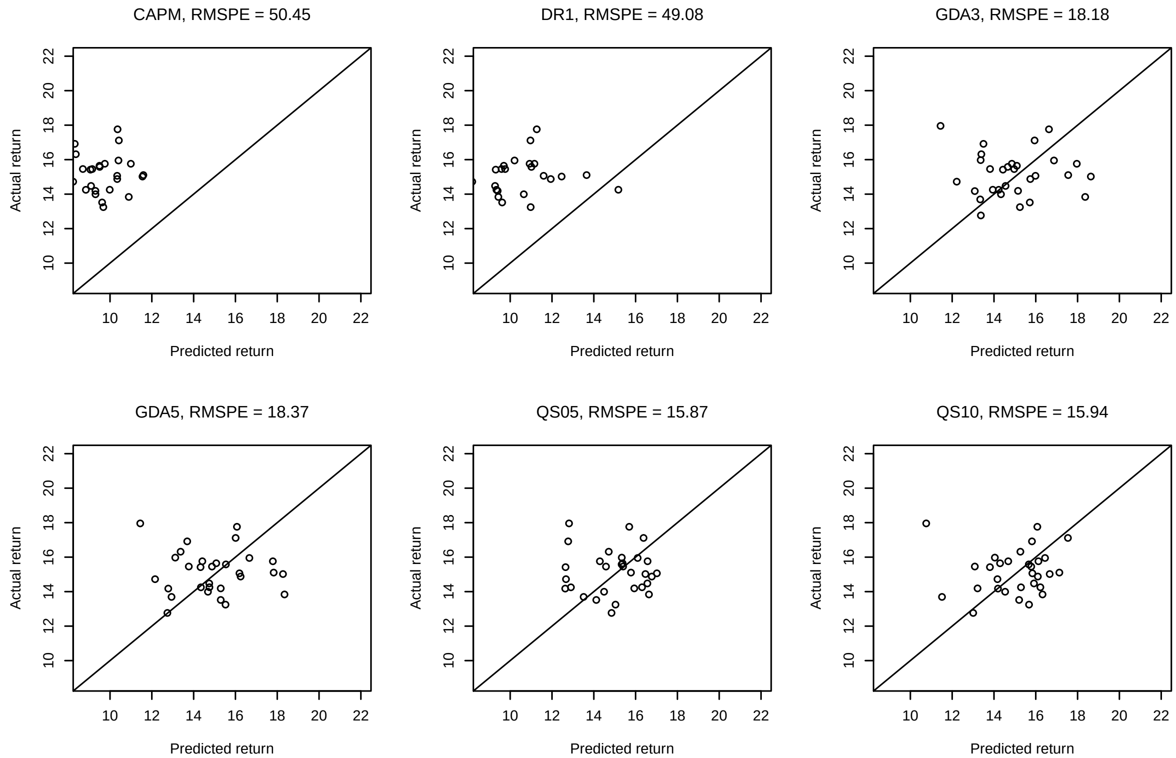

Figure 1: Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 30 Fama-French equal weighted industry portfolios.

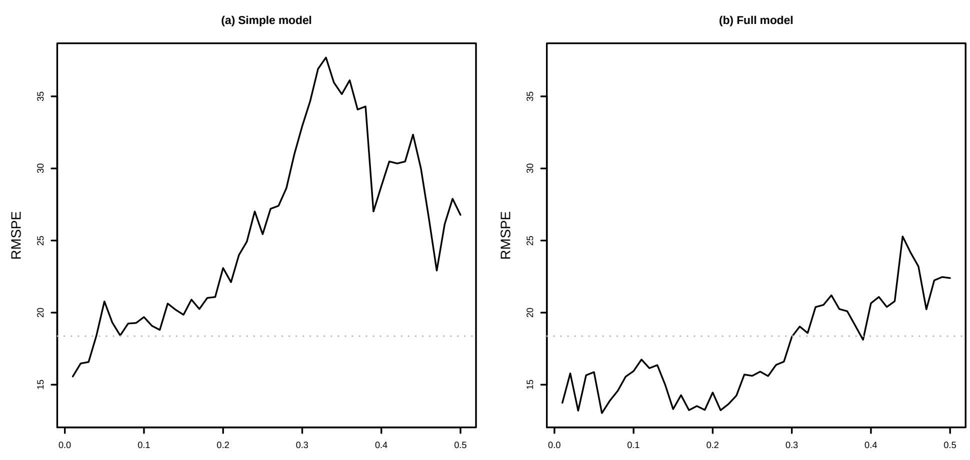

Figure 2: RMSPE for simple and full model estimated on monthly data of 30 Fama-French equal weighted industry portfolios for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model.

τ_{r_m}	Tail market risk					Extreme volatility risk				
	λ^{TR}	λ^{TR}_{long}	λ^{TR}_{short}	λ^{CAPM}	RMSPE	λ^{EVR}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	0.31	0.15	0.90	0.66	14.62	3.85	-0.01	-3.58	0.66	21.35
	(0.41)	(2.05)	(1.45)			(2.45)	(-0.05)	(-2.99)		
0.05	0.53	0.74	0.73	0.66	20.14	0.17	0.43	-3.41	0.66	17.70
	(0.18)	(1.05)	(0.35)			(0.04)	(0.80)	(-0.78)		
0.1	3.09	0.71	-1.64	0.66	19.34	-3.75	0.98	0.73	0.66	16.18
	(0.93)	(0.92)	(-0.59)			(-0.94)	(2.15)	(0.20)		
0.15	8.03	0.14	-5.72	0.66	16.12	-8.88	1.38	4.47	0.66	13.30
	(2.20)	(0.21)	(-1.75)			(-2.17)	(4.40)	(1.15)		
0.25	6.42	1.58	-4.43	0.66	20.63	-1.09	1.00	-3.06	0.66	16.63
	(1.27)	(2.24)	(-0.97)			(-0.19)	(2.80)	(-0.55)		

Table 3: Estimated coefficients from the horse race estimation. Prices of risk of simple 3-factor models also including the simple betas for the respective risks estimated on monthly data of 30 Fama-French equal-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return.

betas based on their t-statistics. We will run the regression separately for the tail market risk betas and extreme volatility risk betas.

The results can be seen in Table 3 and they clearly indicate that the frequency decomposition of risk is a valuable dimension to explore. First, for the TR model we observe that, especially for the lowest quantile, i.e. $\tau_{r_m} = 0.01$, frequency decomposed measures outperform the simple measure of tail market risk. For the models given by the higher quantiles, we observe ambiguous results and cannot clearly decide whether the performance is more driven by the quantile definition of risk only. Moreover, the values of the coefficients (both simple and full model) vary significantly probably because of the correlation between these measures. In that case, we argue that the frequency decomposition is valuable as it is not decisively outperformed by the simple quantile measure. Moreover, the best performance of the model is achieved for $\tau_{r_m} = 0.01$ and in that case the long- and short-term betas drive out the simple beta.

Second, the results of the EVR model are less indecisive. For the low values of the quantiles, the decomposition into horizons is outperformed by the simple measure of extreme volatility risk. On the other hand, with increasing value of quantile, we can see that disentangling the risk into long and short horizon brings a valuable information and moreover, the performance improves in comparison to the low values of τ_{r_m}.

6.6 GDA and QS measures of risk

In this subsection, we compare the performance of our model with the GDA5 model of Farago and Tédongap (2017). To do that, we construct horse race regressions between their measures of risk and ours. We compare risk measures associated with market return and market volatility increments separately. The aim of this analysis is to decide which measures of risk better capture the notion of extreme risks associated with risk premium.

The results are depicted in Table 4. In case of tail market risk, we see that GDA5 measures of risk (λ_D and λ_{WD}) do not drive out our measures of risk for any value of τ_{r_m}. This clearly suggests that our measures are better in capturing the asymmetric risk priced in the cross-section of assets.

In case of volatility risk, we see from Table 4 that the situation is pretty much the same as in the case of tail market risk. Especially, the price of risk for long-term EVR betas stays strong over all values of quantile, and GDA5 measures of volatility risk remain insignificant in all of the cases.

All the results suggest that our model brings an improvement in terms of identifying form of asymmetric risk which is priced in the cross-section of asset returns.

	Tail market risk						Extreme volatility risk					
τ_{r_m}	λ_D	λ_{WD}	λ^{TR}_{long}	λ^{TR}_{short}	λ^{CAPM}	RMSPE	λ_X	λ_{XD}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	-0.03	0.04	-0.14	0.89	0.66	12.42	2.73	-1.12	0.17	-0.84	0.66	22.26
	(-0.69)	(0.12)	(-0.97)	(4.66)			(1.21)	(-0.57)	(1.85)	(-1.30)		
0.05	-0.02	0.14	-0.14	1.25	0.66	17.08	-1.73	1.36	0.48	-3.72	0.66	17.44
	(-0.36)	(0.36)	(-0.39)	(2.28)			(-0.87)	(0.86)	(4.54)	(-2.83)		
0.1	0.01	0.28	0.33	0.98	0.66	17.28	-1.98	1.34	0.63	-3.22	0.66	16.07
	(0.19)	(0.72)	(0.71)	(1.41)			(-1.07)	(0.89)	(5.77)	(-2.43)		
0.15	-0.02	-0.03	0.93	1.01	0.66	16.17	0.07	-0.51	0.78	-3.69	0.66	14.21
	(-0.45)	(-0.07)	(2.40)	(1.30)			(0.05)	(-0.42)	(7.02)	(-3.32)		
0.25	-0.03	0.04	1.02	0.66	0.66	16.90	1.56	-1.10	0.85	-4.14	0.66	16.34
	(-0.44)	(0.10)	(2.09)	(0.72)			(0.96)	(-0.79)	(5.42)	(-2.89)		

Table 4: Estimated coefficients from the horse race estimation between QS measures of risk and GDA5 measures of risk. Prices of risk of 3-factor models including respective GDA5 measures of risk estimated on monthly data of 30 Fama-French equal-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return.

6.7 Robustness checks

As robustness check, we first report results based on 30 Fama-French industry portfolio data which are value weighted. Results are summarized in Appendix B.1. We report estimated coefficients for both simple and full model, RMSPE for continuum of τ_{r_m} and comparison with competing models. We also conduct the same analysis with volatility being computed from daily data as a realized volatility for each month in the sample. It is obvious from estimated simple models that both tail market risk and extreme volatility risk are priced in cross-section. Estimated full models suggest that short tail risk is the driving force of aggregated tail risk, and although coefficients for long extreme volatility risk are not significant, they posses the right sign and are numerically close to their counterparts computed on volatility from GARCH model. We argue that this is due to highly non-smoothed nature of the volatility computed as a sum over respective months. On the other hand, EVR is consistently priced using the Simple 3-factor model. This seems natural as the realized volatility poses non-smoothed nature and the frequency decomposition is not so effective as in the case of smoothed volatility estimates as in the case of GARCH model.

In Appendix D, we perform the same analysis on 25 Fama-French portfolios sorted on size and book-to-market. We report results based on both equal and value weighted portfolios, and volatility is computed using GARCH model and as realized volatility from daily data. In the case of models with volatility computed from GARCH model, our model performs comparable to GDA5 model but slightly worse, but outperforms all the other competing models, and moreover all the features observed in the case of 30 industry portfolios are present in this case, also, with values of the coefficients being similar. In the case of volatility computed from daily data, our model outperforms all the competing models including GDA5 model.

As an another robustness check for the TR betas, in the first stage regression, we standardize the returns of both market and portfolio returns using estimated volatility and estimate the TR betas using these transformed series, and the second stage regression remains the same. Volatility is estimated for each time series separately using GARCH(1,1) model for simplicity reasons. This robustness check aims to show that the betas do not solely capture the common

τ_{r_m}	Simple model λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	Full model λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	0.11	0.24	0.03	1.02	-0.02	0.16	0.03	0.26	0.03	1.02
	(3.50)	(2.66)			(-0.81)	(3.14)	(2.10)	(2.19)		
0.05	0.16	0.28	0.03	1.01	-0.01	0.19	0.03	0.28	0.03	1.02
	(3.84)	(4.05)			(-0.27)	(2.23)	(2.13)	(1.32)		
0.1	0.20	0.33	0.03	1.07	-0.02	0.24	0.03	0.19	0.03	1.04
	(4.34)	(4.75)			(-0.35)	(3.00)	(2.61)	(0.90)		
0.15	0.22	0.46	0.03	1.25	-0.02	0.27	0.05	0.02	0.03	1.15
	(3.85)	(5.43)			(-0.39)	(3.46)	(3.75)	(0.09)		
0.25	0.27	1.01	0.03	1.73	-0.11	0.40	0.06	0.59	0.03	1.37
	(3.37)	(7.50)			(-1.58)	(4.19)	(4.21)	(1.39)		

Table 5: Estimated coefficients. Prices of risk of Simple 3-factor and Full 5-factor model estimated on daily data of 30 Fama-French equal-weighted industry portfolios sampled between July 1926 and March 2019. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return.

trend in volatility present in both market and portfolio returns. Results of this analysis are captured in Table 13. We observe that, especially for the long-term TR betas, the coefficients remain significant even after this standardization procedure.

6.8 Estimation on daily data

We estimate our Simple and Full model using daily data, also. First, we estimate our models on 30 Fama-French equal-weighted industry portfolios and compare the results with the competing models. We know that the performance of asset pricing models, in most of the cases, is substantially worse when working with data with frequency higher than one month and some of them are even useless in this case (e.g., CAPM model). We want show that this is not the case of our models and that the models perform better in this case than the other models

Estimated parameters of our Simple and Full model are summarized in Table 5. We can see that the coefficients for Simple model are significant through all the quantiles but the best performance is achieved for low values of τ_{r_m}. The comparison between our models and the other is depicted in Figure 3 and based on RMSPE, our model predicts the returns the best among all the models in our investigation. Performance of the model in relation to τ_{r_m} is captured in Figure 4. We can see that our models achieve better performance that the GDA5 model for low values of τ_{r_m}.

In the Appendix E, we report also estimates of the models on daily data of 25 Fama-French equal weighted portfolios sorted on size and book-to-market. We choose this dataset for the analysis because our models in the robustness check procedure perform relatively worst on this dataset when compared to the GD5 model. So, to see whether this also holds for the daily data, we compare all the competing models on this dataset. The estimated coefficients of the Simple 3-factor and Full 5-factor model are reported in Table 14. From the comparisons between the competing models depicted in Figures 20 and 19, we can see that our models provide a significantly better fit to the daily data, and the Full 5-factor model achieves better performance for all values of τ.

To summarize, we can see that the both in terms of coefficient significancy and relative model performance, our models can provide a good fit to the daily data, which is quite uncommon among asset pricing models. This results only confirms our hypothesis that the extreme risks are priced in the cross-section of asset returns.

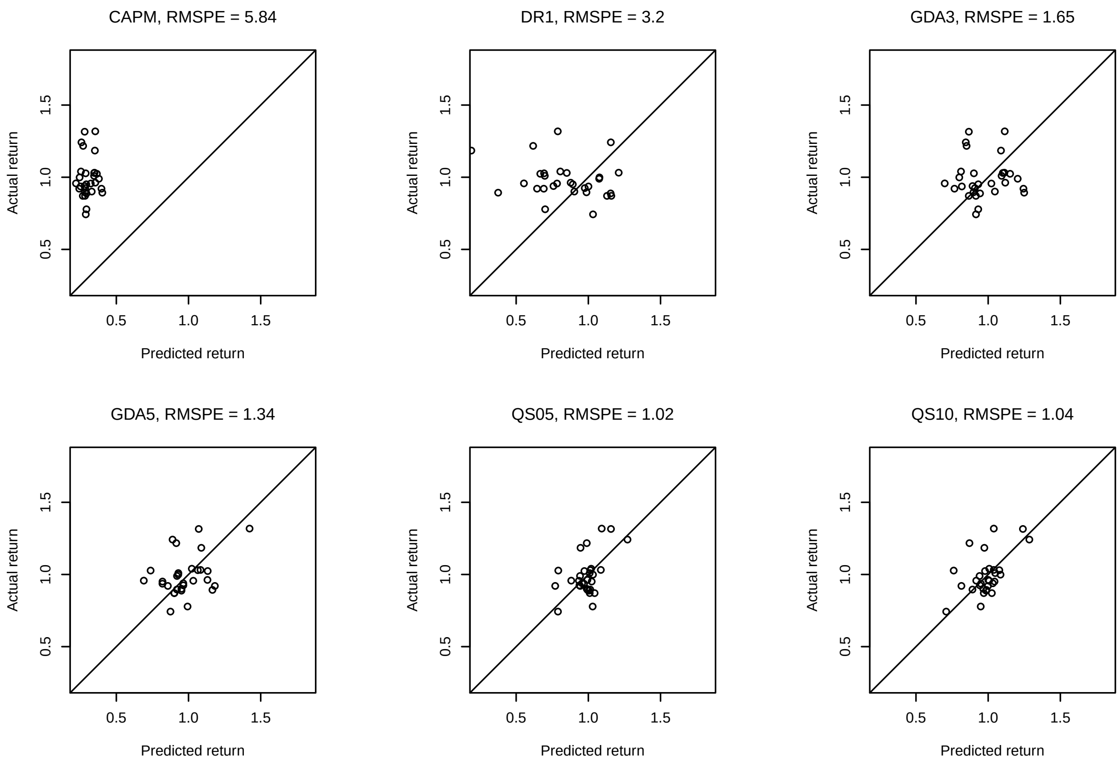

Figure 3: Predicted returns. Plots of predicted versus actual returns for competing models estimated on daily data of 30 Fama-French equal-weighted industry portfolios.

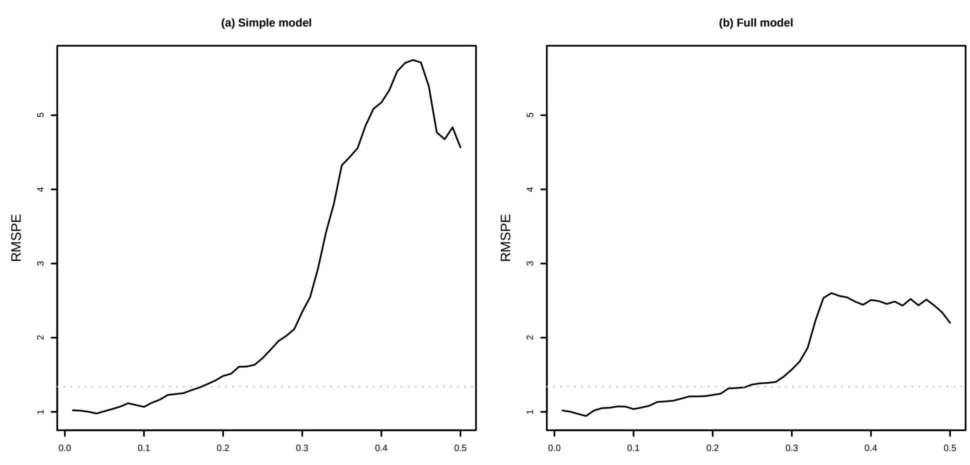

Figure 4: RMSPE for simple and full model estimated on daily data of 30 Fama-French equal-weighted industry portfolios. Horizontal line represents RMSPE of GDA5 model.

7 Conclusion

We have shown that extreme risks are priced in cross-section of asset returns. In the paper, we distinguish between tail market risk and extreme volatility risk. Tail market risk is characterized by the dependence between highly negative market and asset events. Extreme volatility risk is defined as cooccurrence of extremely high increases of market volatility and highly negative asset returns. Negative events are derived from distribution of market returns and its respective quantile is used for determining threshold values for computing quantile cross-spectral betas. We define two empirical models for testing associated risk premium. Simple model, which does not take into consideration frequency aspect, confirms that investors require premium for bearing both tail market risk and extreme volatility risk. Full model further identifies that premium for tail market risk is mostly featured in its short-term component, and premium for extreme volatility risk is mostly associated with its long-term component.

In order to consistently estimate the model, data with long enough history has to be employed. But if the data are available, our model is able to outperform competing models and its performance is best for low threshold values suggesting that investors require risk premium for holding assets susceptible to extreme risks. Moreover, our models can perform very well even on the daily data, which is not common for asset pricing models.

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A Estimation of quantile cross-spectral betas

Estimation of QS betas defined in our paper is based on the smoothed quantile cross-periodograms studied in Baruník and Kley (2015). For a strictly stationary time series $X_{0,j}, \ldots, X_{n-1,j}$, we define $I\{\hat{F}_{n,j}(X_{t,j}) \leq \tau\} = I\{R_{n;t,j} \leq n\tau\}$ where $\hat{F}_{n,j}(x) \equiv n^{-1} \sum_{t=0}^{n-1} I\{X_{t,j} \leq x\}$ is the empirical distribution function of $X_{t,j}$ and $R_{n;t,j}$ denotes the rank of $X_{t,j}$ among $X_{0,j}, \ldots, X_{n-1,j}$. We have seen that the cornerstone of quantile cross-spectral beta is quantile cross-spectral density defined in Equation 52. Its population counterpart is called rank-based copula cross-periodogram, CCR-periodogram, and is defined as

$$I_{n,R}^{j_1,j_2}(\omega; \tau_1, \tau_2) \equiv \frac{1}{2\pi n} d_{n,R}^{j_1}(\omega; \tau_1) d_{n,R}^{j_2}(-\omega; \tau_2) \tag{68}$$

where

$$d_{n,R}^{j}(\omega; \tau) \equiv \sum_{t=0}^{n-1} I\{\hat{F}_{n,j}(X_{t,j}) \leq \tau\} e^{-i\omega t} = \sum_{t=0}^{n-1} I\{R_{n;t,j} \leq n\tau\} e^{-i\omega t}, \quad \tau \in [0,1]. \tag{69}$$

As discussed in Baruník and Kley (2015), CCR-periodogram is not a consistent estimator of quantile cross-spectral density. Consistency can be achieved by smoothing CCR-periodogram across frequencies. Following Baruník and Kley (2015), we employ the following

$$\hat{G}_{n,R}^{j_1,j_1}(\omega; \tau_1, \tau_2) \equiv \frac{2\pi}{n} \sum_{s=0}^{n-1} W_n(\omega - 2\pi s/n) I_{n,R}^{j_1,j_2}(2\pi s/n, \tau_1, \tau_2) \tag{70}$$

where W_n is defined in Section 3 of Baruník and Kley (2015). Estimator of quantile cross-spectral beta is defined as

$$\hat{\beta}_{n,R}^{j_1,j_2}(\omega; \tau_1, \tau_2) \equiv \frac{\hat{G}_{n,R}^{j_1,j_2}(\omega; \tau_1, \tau_2)}{\hat{G}_{n,R}^{j_2}(\omega; \tau_2)}. \tag{71}$$

Consistency of the estimator can be proven using exactly same logic as in Theorem 3.4 in Baruník and Kley (2015) by replacing quantile coherency with quantile cross-spectral beta.

B Robustness checks

B.1 Realized volatility

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.02	0.43	0.66	12.94	-0.08	0.85	0.07	0.55	0.66	12.30
	(5.36)	(3.67)			(-0.48)	(3.66)	(0.46)	(1.87)		
0.05	1.22	0.57	0.66	18.66	-0.21	1.40	0.42	0.11	0.66	18.20
	(2.12)	(2.67)			(-0.38)	(2.22)	(0.98)	(0.17)		
0.1	0.72	0.91	0.66	16.28	0.05	0.80	0.25	0.54	0.66	16.43
	(1.18)	(4.58)			(0.09)	(1.01)	(0.57)	(0.73)		
0.15	1.59	0.91	0.66	15.80	0.73	0.70	-0.07	0.87	0.66	15.41
	(2.05)	(4.67)			(1.64)	(0.75)	(-0.15)	(1.03)		
0.25	1.32	1.43	0.66	15.84	0.23	1.02	0.49	0.45	0.66	15.27
	(1.65)	(7.64)			(0.41)	(1.15)	(1.56)	(0.87)		

Table 6: Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 30 Fama-French equal weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

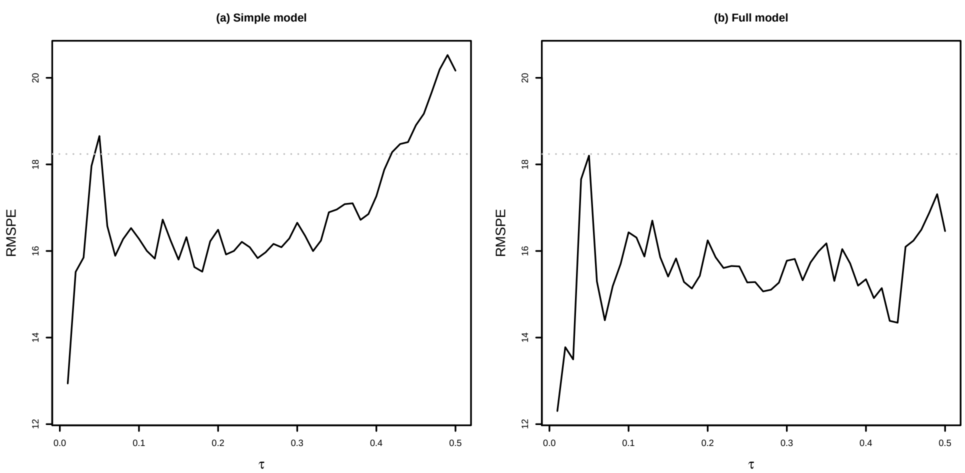

Figure 5: RMSPE for simple and full model estimated on monthly data of 30 Fama-French equal weighted industry portfolios for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model. Volatility is computed as realized volatility from daily data.

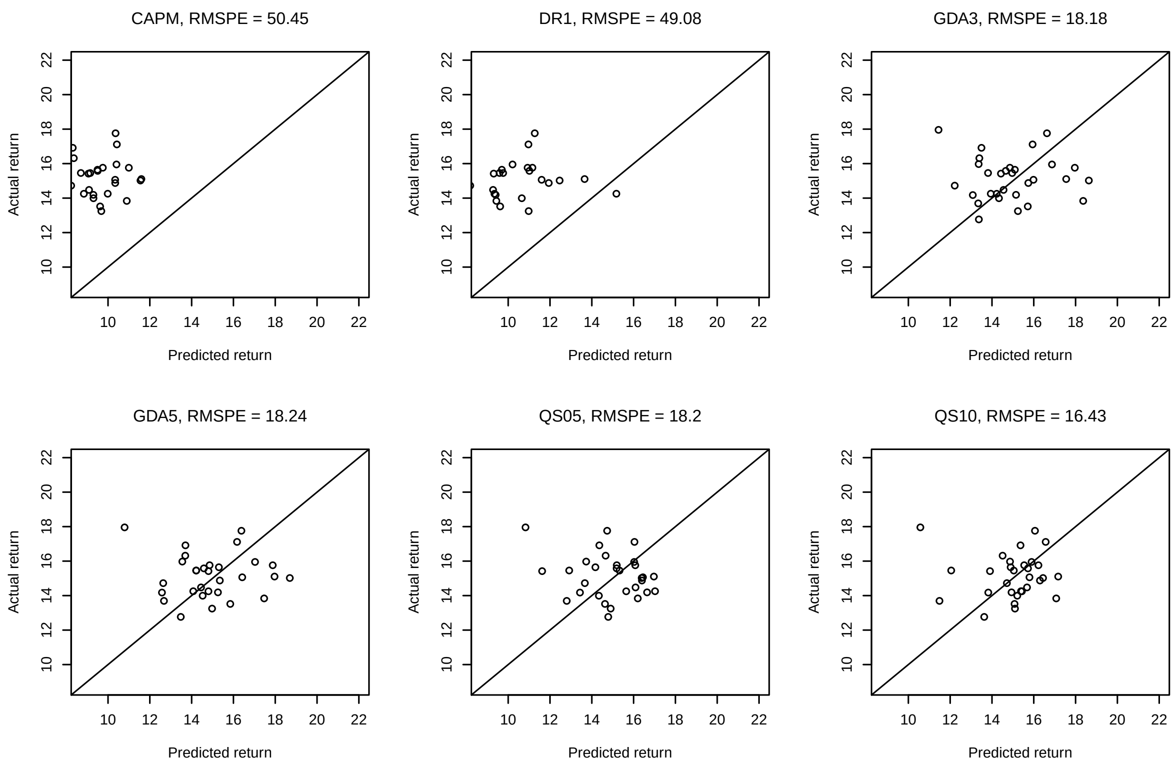

Figure 6: Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 30 Fama-French equal weighted industry portfolios. Volatility is computed as realized volatility from daily data.

B.2 Value weighted portfolios

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.36	-0.50	0.66	16.72	-0.03	1.06	0.06	-0.34	0.66	16.52
	(9.88)	(-1.01)			(-0.15)	(6.17)	(0.34)	(-0.73)		
0.05	2.21	0.29	0.66	15.65	-0.12	2.12	0.18	0.62	0.66	14.92
	(5.58)	(0.57)			(-0.27)	(5.25)	(1.22)	(0.67)		
0.1	1.69	1.77	0.66	19.84	-0.54	1.93	0.39	-0.57	0.66	18.01
	(2.84)	(3.09)			(-1.11)	(3.16)	(3.51)	(-0.37)		
0.15	1.79	3.13	0.66	19.56	0.05	1.42	0.36	0.32	0.66	18.09
	(3.10)	(4.09)			(0.12)	(2.00)	(2.95)	(0.18)		
0.25	2.37	3.93	0.66	28.48	0.08	1.40	0.57	-2.39	0.66	18.19
	(2.33)	(2.52)			(0.17)	(1.85)	(4.88)	(-1.71)		

Table 7: Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 30 Fama-French value weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return.

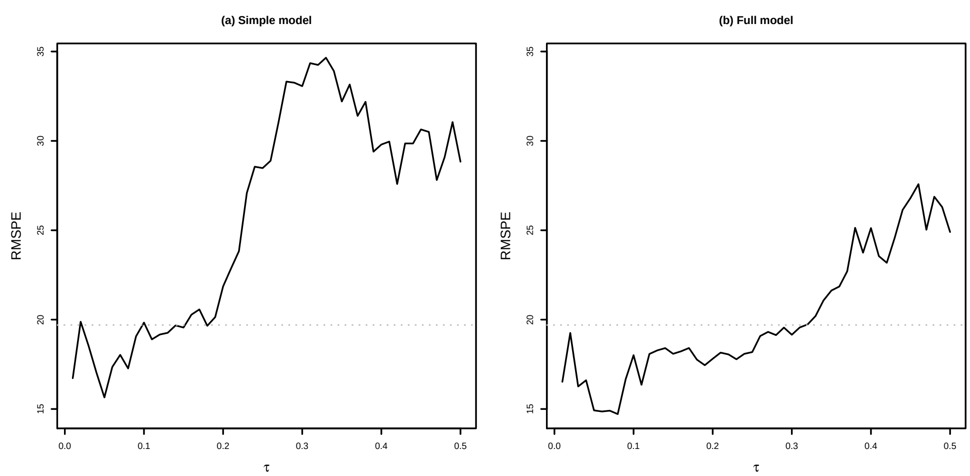

Figure 7: RMSPE for simple and full model estimated on monthly data of 30 Fama-French value weighted industry portfolios for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model.

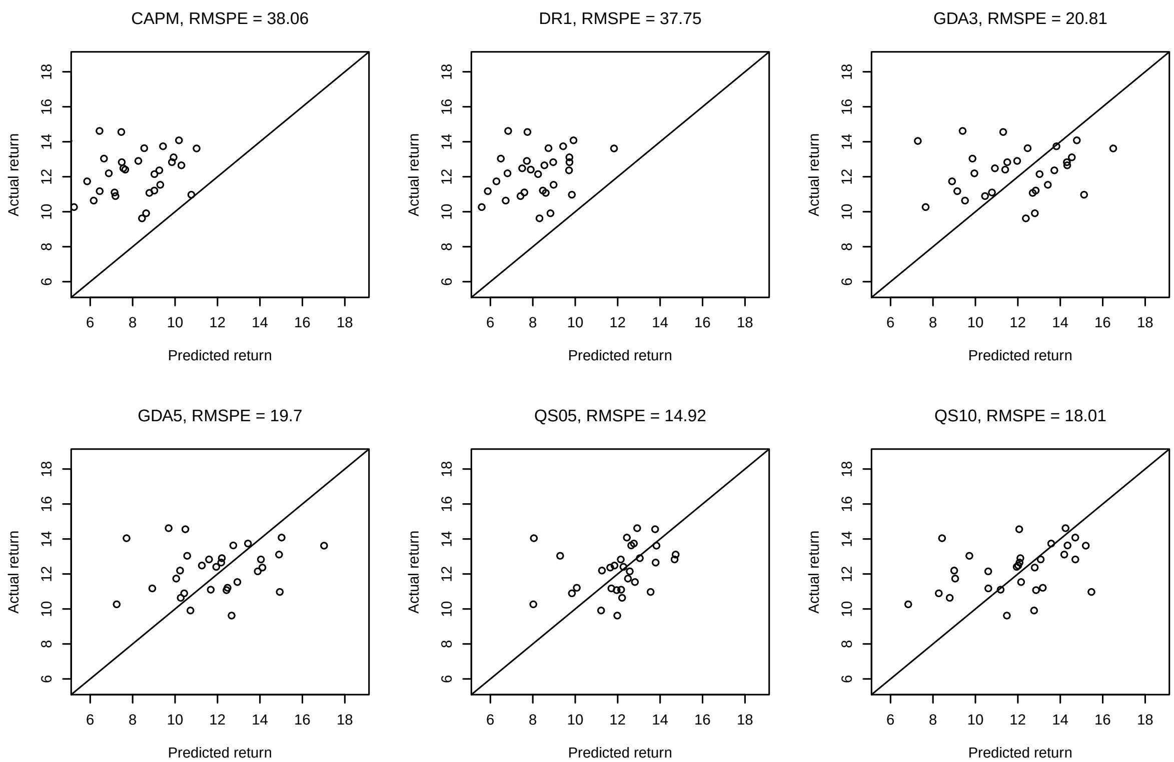

Figure 8: Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 30 Fama-French value weighted industry portfolios.

B.3 Value weighted portfolios and realized volatility

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.12	0.11	0.66	16.87	0.22	0.33	-0.32	1.09	0.66	15.01
	(4.14)	(0.71)			(0.81)	(0.99)	(-1.41)	(2.53)		
0.05	2.31	0.04	0.66	15.73	-0.22	2.15	0.23	0.05	0.66	15.08
	(4.87)	(0.21)			(-0.44)	(4.23)	(0.69)	(0.10)		
0.1	1.36	0.56	0.66	18.91	-0.84	2.44	0.73	-0.42	0.66	17.53
	(2.28)	(3.65)			(-1.41)	(2.92)	(1.42)	(-0.46)		
0.15	1.40	0.71	0.66	18.35	0.09	1.44	0.22	0.39	0.66	18.07
	(2.43)	(4.78)			(0.15)	(1.62)	(0.38)	(0.36)		
0.25	0.94	1.18	0.66	17.79	-0.10	1.42	0.43	0.41	0.66	17.31
	(1.44)	(7.75)			(-0.21)	(1.55)	(1.10)	(0.60)		

Table 8: Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 30 Fama-French value weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

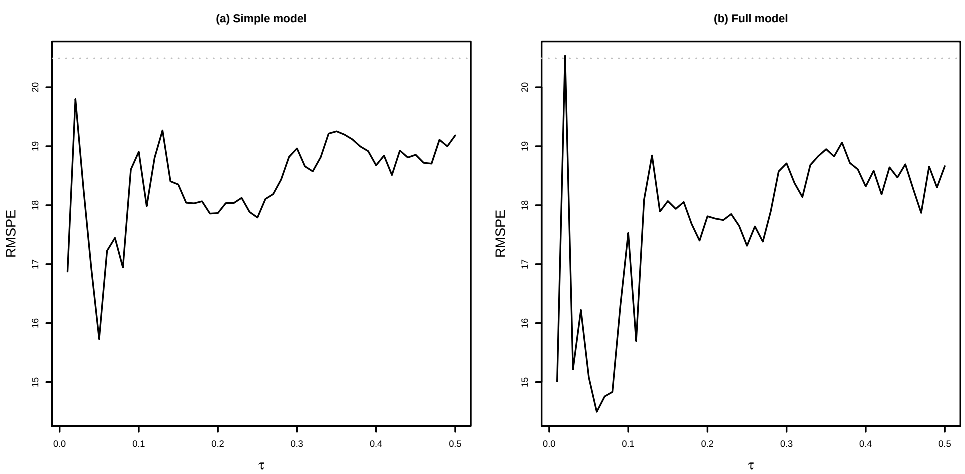

Figure 9: RMSPE for simple and full model estimated on monthly data of 30 Fama-French value weighted industry portfolios for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model. Volatility is computed as realized volatility from daily data.

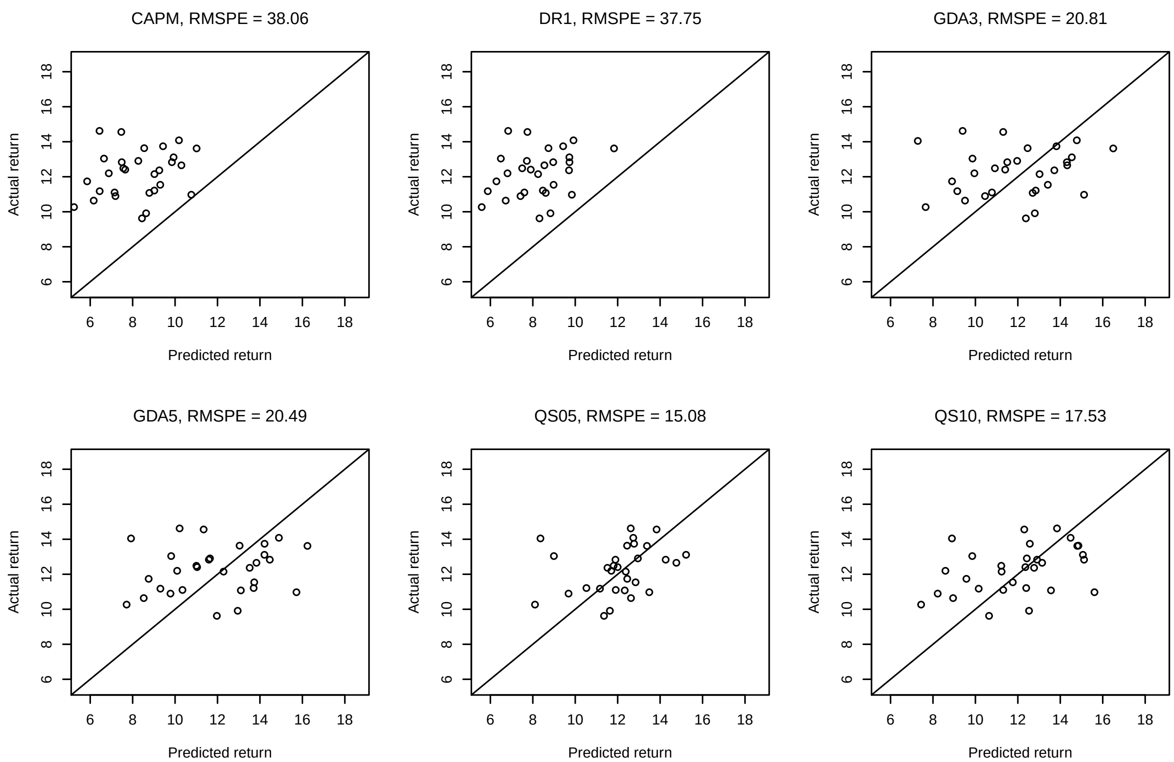

Figure 10: Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 30 Fama-French value weighted industry portfolios. Volatility is computed as realized volatility from daily data.

C Results for 25 F-F portfolios sorted on size and book-to-market

C.1 Equal weighted portfolios

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.81	-0.05	0.66	28.28	-0.25	0.86	0.38	0.20	0.66	22.80
	(5.14)	(-0.05)			(-0.72)	(1.99)	(1.42)	(0.27)		
0.05	3.66	0.10	0.66	21.15	-0.30	2.84	0.32	-1.93	0.66	20.60
	(5.40)	(0.13)			(-0.60)	(1.90)	(1.85)	(-1.30)		
0.1	-0.09	3.52	0.66	21.97	0.53	-0.23	0.21	6.12	0.66	21.25
	(-0.10)	(4.59)			(0.87)	(-0.15)	(1.35)	(1.90)		
0.15	0.32	4.80	0.66	22.64	-0.39	1.04	0.40	6.58	0.66	22.17
	(0.25)	(3.93)			(-0.41)	(0.42)	(1.80)	(2.39)		
0.25	-2.46	13.08	0.66	22.16	-1.49	-0.39	0.92	11.29	0.66	19.12
	(-1.31)	(4.58)			(-1.91)	(-0.17)	(6.22)	(3.57)		

Table 9: Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 25 Fama-French equal weighted portfolios sorted on size and book-to-market sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return.

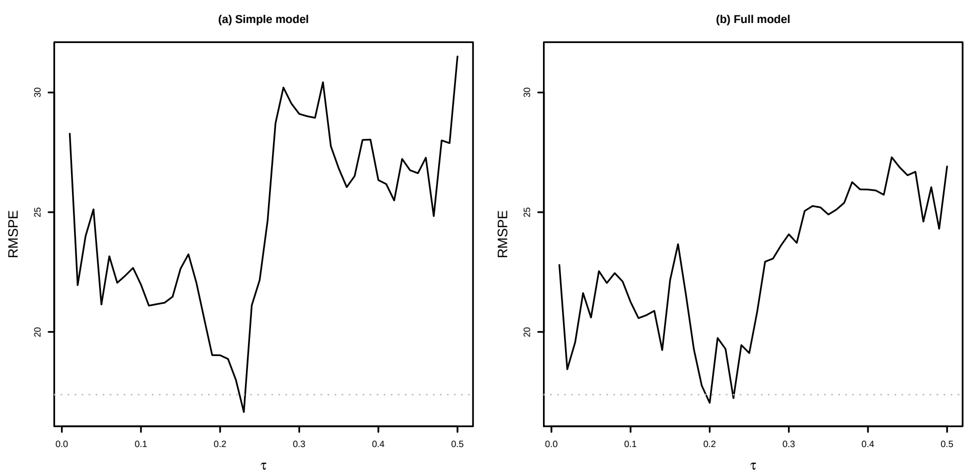

Figure 11: RMSPE for simple and full model estimated on monthly data of 25 Fama-French equal weighted portfolios sorted on size and book to market for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model.

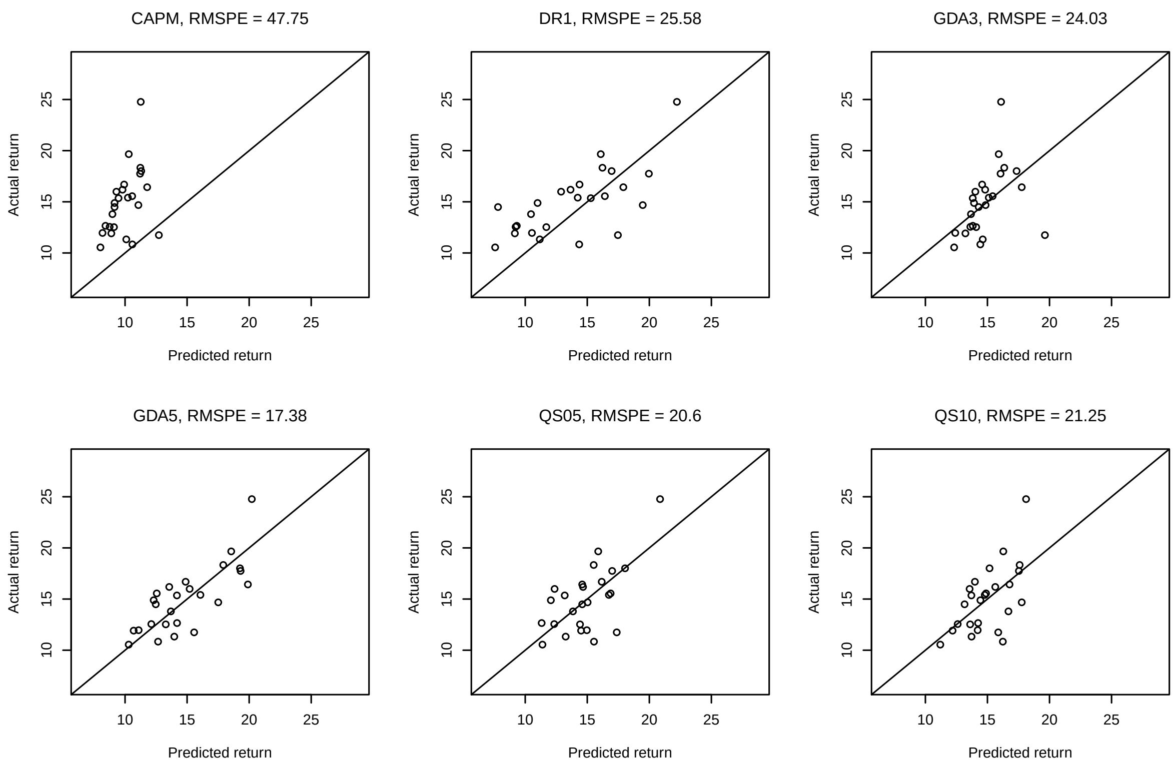

Figure 12: Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 25 Fama-French equal weighted portfolios sorted on size and book to market.

C.2 Realized volatility

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	0.58	0.67	0.66	24.41	-0.25	1.18	0.40	-0.19	0.66	22.64
	(1.16)	(2.81)			(-0.71)	(1.63)	(1.23)	(-0.32)		
0.05	3.00	0.18	0.66	20.93	-0.16	3.42	0.00	0.73	0.66	20.77
	(2.69)	(0.69)			-0.26	(2.18)	(0.00)	(0.65)		
0.1	-0.32	1.01	0.66	23.55	1.46	-3.41	-0.96	2.43	0.66	22.03
	(-0.28)	(3.92)			(1.50)	(-1.41)	(-1.01)	(1.49)		
0.15	1.00	0.91	0.66	23.49	-1.39	4.03	1.46	-1.22	0.66	21.66
	(0.83)	(3.56)			(-1.47)	(1.63)	(2.07)	(-0.89)		
0.25	0.54	1.28	0.66	23.88	-1.42	2.79	1.79	-1.75	0.66	20.59
	(0.36)	(3.86)			(-1.73)	(1.30)	(2.68)	(-1.45)		

Table 10: Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 25 Fama-French equal weighted portfolios sorted on size and book-to-market sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

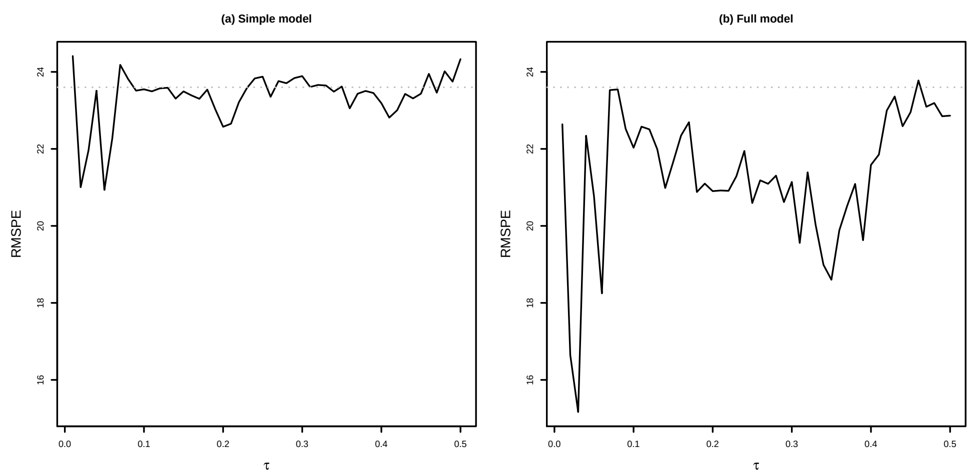

Figure 13: RMSPE for simple and full model estimated on monthly data of 25 Fama-French equal weighted portfolios sorted on size and book to market for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model. Volatility is computed as realized volatility from daily data.

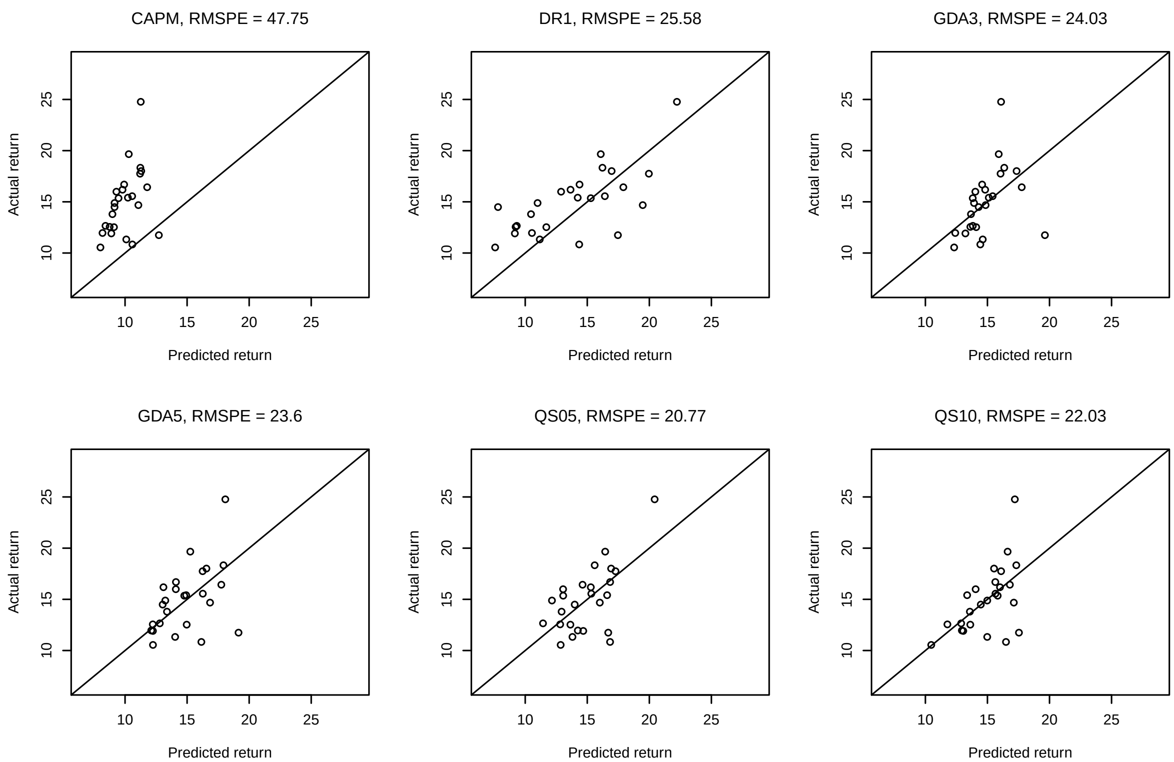

Figure 14: Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 25 Fama-French equal weighted portfolios sorted on size and book to market. Volatility is computed as realized volatility from daily data.

C.3 Value weighted portfolios

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.59	0.74	0.66	24.39	-0.45	0.58	0.50	0.41	0.66	18.81
	(5.18)	(0.92)			(-1.51)	(1.43)	(2.44)	(0.68)		
0.05	2.90	0.84	0.66	21.56	-0.87	0.57	0.59	-2.69	0.66	19.25
	(2.38)	(0.70)			(-2.02)	(0.40)	(3.55)	(-1.74)		
0.1	-1.55	4.04	0.66	18.40	-0.03	-3.03	0.51	-2.21	0.66	16.89
	(-1.93)	(6.67)			(-0.05)	(-2.54)	(4.70)	(-0.87)		
0.15	-1.89	5.98	0.66	20.02	-0.81	-1.25	0.55	4.87	0.66	19.32
	(-1.65)	(5.57)			(-1.08)	(-0.82)	(3.46)	(1.56)		
0.25	-0.32	8.54	0.66	24.33	-1.26	-0.58	0.86	3.69	0.66	19.55
	(-0.26)	(4.38)			(-1.59)	(-0.38)	(5.59)	(1.47)		

Table 11: Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 25 Fama-French value weighted portfolios sorted on size and book-to-market sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return.

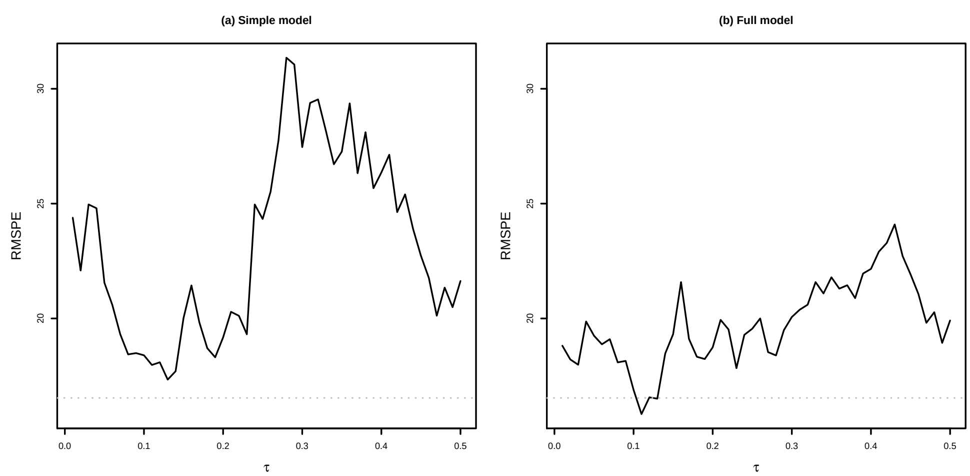

Figure 15: RMSPE for simple and full model estimated on monthly data of 25 Fama-French value weighted portfolios sorted on size and book to market for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model.

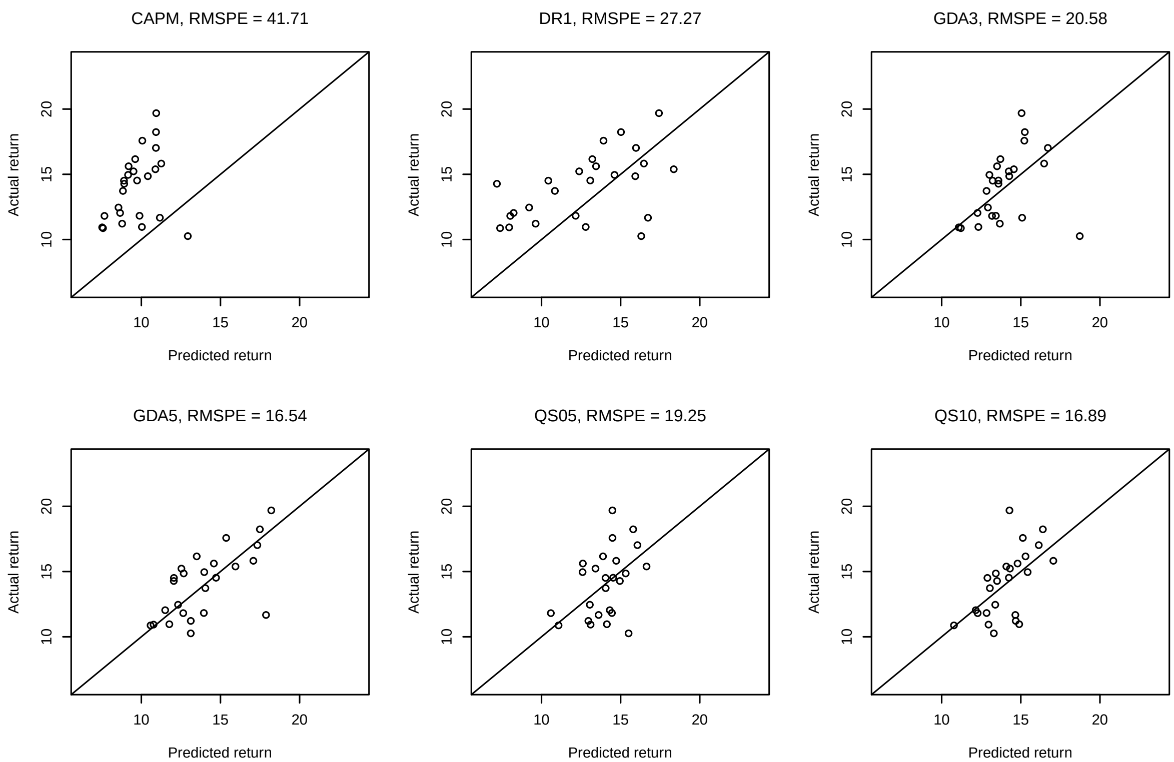

Figure 16: Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 25 Fama-French value weighted portfolios sorted on size and book to market.

C.4 Value weighted portfolios and realized volatility

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	0.34	0.68	0.66	21.38	-0.68	1.67	0.84	-0.94	0.66	15.49
	(0.63)	(2.83)			(-3.02)	(3.23)	(4.18)	(-2.27)		
0.05	1.31	0.49	0.66	20.74	-0.86	0.21	0.52	0.39	0.66	19.83
	(0.81)	(1.54)			(-1.47)	(0.14)	(1.08)	(0.47)		
0.1	-2.53	1.30	0.66	17.72	0.05	-2.96	0.18	0.80	0.66	17.06
	(-2.87)	(7.06)			(0.07)	(-2.31)	(0.29)	(0.76)		
0.15	-1.47	1.23	0.66	19.44	-1.43	0.52	1.60	-1.47	0.66	16.68
	(-1.42)	(5.85)			(-2.10)	(0.36)	(2.88)	(-1.49)		
0.25	-1.80	1.59	0.66	18.59	-1.42	-0.37	1.26	-0.57	0.66	15.86
	(-1.79)	(7.02)			(-2.22)	(-0.29)	(2.71)	(-0.67)		

Table 12: Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 25 Fama-French value weighted portfolios sorted on size and book-to-market sampled between July 1927 and November 2017. Model is estimated for various values of thresholds given by τ_{r_m}. Market price of risk is imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

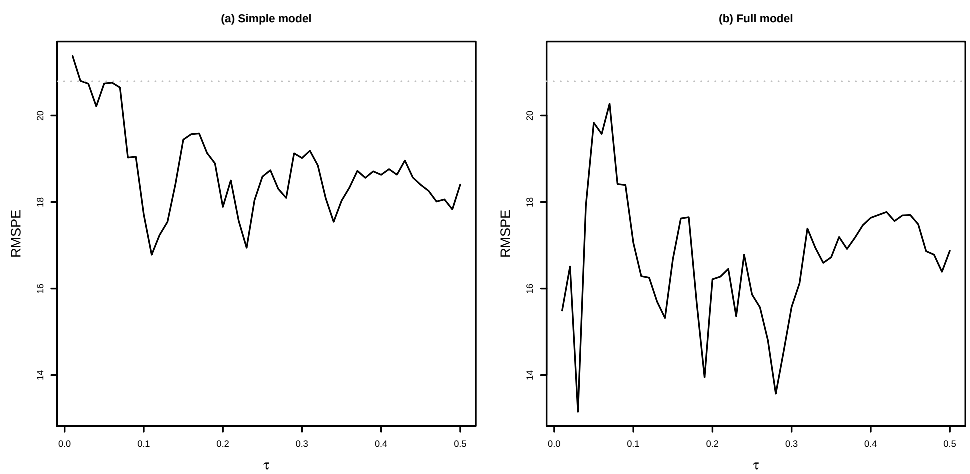

Figure 17: RMSPE for simple and full model estimated on monthly data of 25 Fama-French equal weighted portfolios sorted on size and book to market for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model. Volatility is computed as realized volatility from daily data.

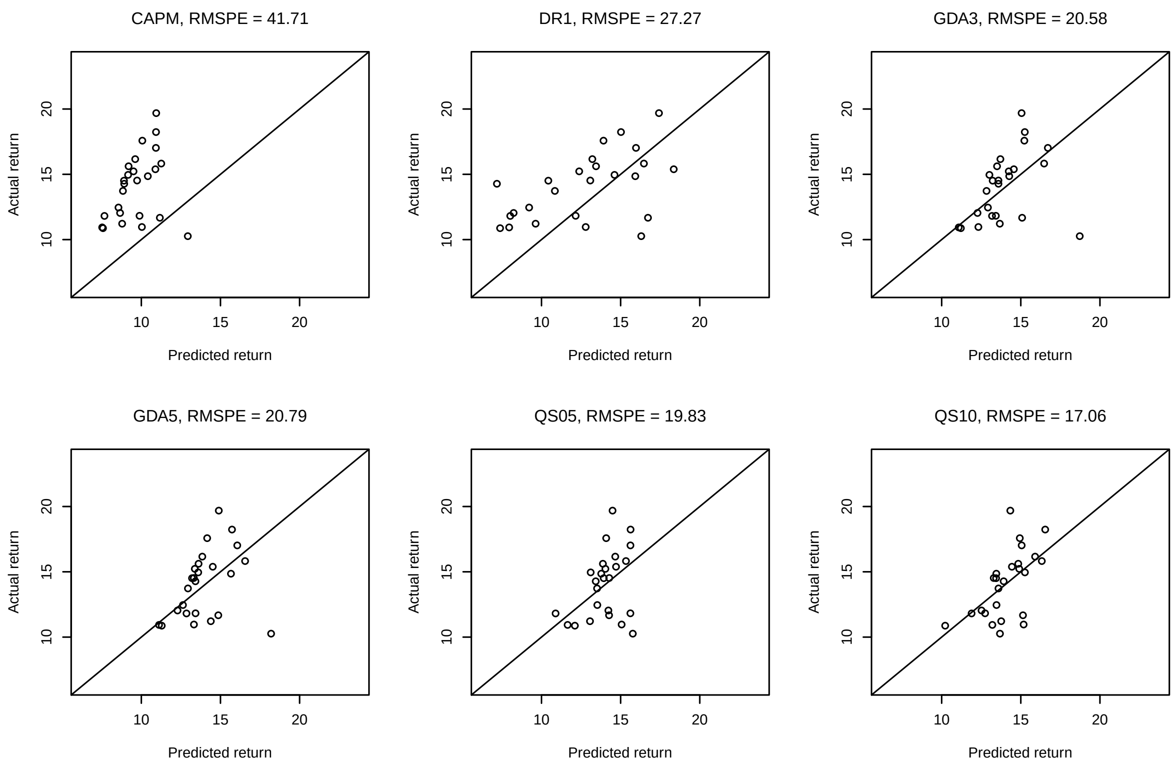

Figure 18: Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 25 Fama-French value weighted portfolios sorted on size and book to market. Volatility is computed as realized volatility from daily data.

D Results for the standardized returns

τ_{r_m}	λ^{TR}_{long}	λ^{TR}_{short}	λ^{CAPM}	RMSPE
0.01	0.82	0.50	0.66	16.55
	(2.64)	(1.19)		
0.05	2.01	1.12	0.66	30.02
	(2.37)	(0.87)		
0.1	0.91	2.54	0.66	23.94
	(2.00)	(3.34)		
0.15	2.41	0.42	0.66	23.40
	(6.05)	(0.38)		
0.25	2.38	0.67	0.66	27.63
	(4.48)	(0.43)		

Table 13: Estimated parameters of the TR 3-factor model. Betas are estimated on 30 Fama-French equal weighted standardized portfolios and market returns. For each series, GARCH(1,1) model is estimated and returns are divided by the estimated conditional volatility.

E Daily data: esults for 25 F-F portfolios sorted on size and book-to-market

E.1 Equal weighted portfolios

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	0.31	-0.20	0.03	2.34	-0.03	0.28	0.00	0.42	0.03	1.95
	(2.29)	(-0.73)			(-0.75)	(2.36)	(0.20)	(1.35)		
0.05	0.29	0.02	0.03	2.56	-0.00	0.12	-0.01	1.21	0.03	2.17
	(2.99)	(0.17)			(-0.03)	(0.57)	(-0.50)	(2.87)		
0.1	0.28	0.12	0.03	2.60	-0.01	0.21	-0.01	1.29	0.03	2.21
	(3.01)	(0.83)			(-0.07)	(0.69)	(-0.54)	(2.23)		
0.15	0.31	0.19	0.03	2.47	0.11	0.10	-0.03	0.86	0.03	2.15
	(3.42)	(1.24)			(0.51)	(0.40)	(-1.47)	(1.04)		
0.25	0.31	0.70	0.03	2.24	0.08	0.18	-0.02	1.10	0.03	2.07
	(3.26)	(3.17)			(0.40)	(1.05)	(-1.62)	(0.88)		

Table 14: Estimated coefficients. Prices of risk of Simple 3-factor and Full 5-factor model estimated on daily data of 25 Fama-French equal-weighted portfolios sorted on size and book-to-market. Model is estimated for various values of thresholds. Market price of risk is imposed to be equal to the average market return.

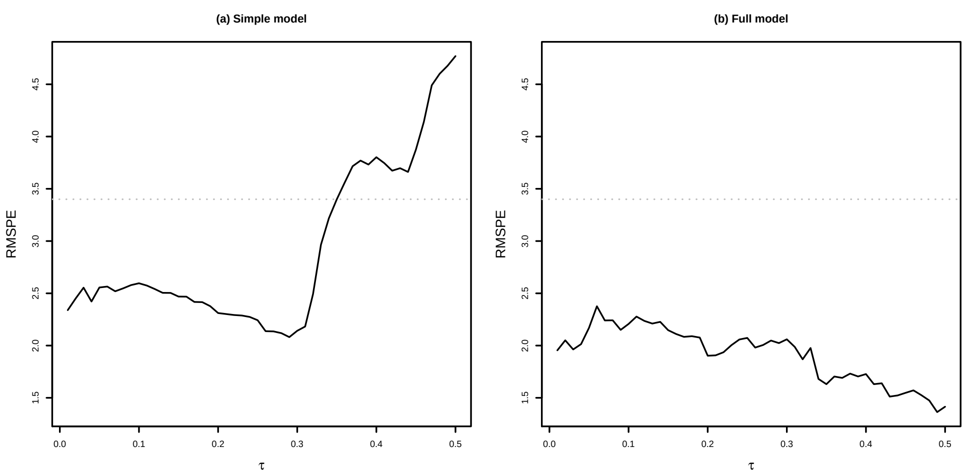

Figure 19: RMSPE for simple and full model estimated on daily data of 30 Fama-French equal weighted industry portfolios for various values of threshold given by τ quantile of market returns. Horizontal line represents RMSPE of GDA5 model.

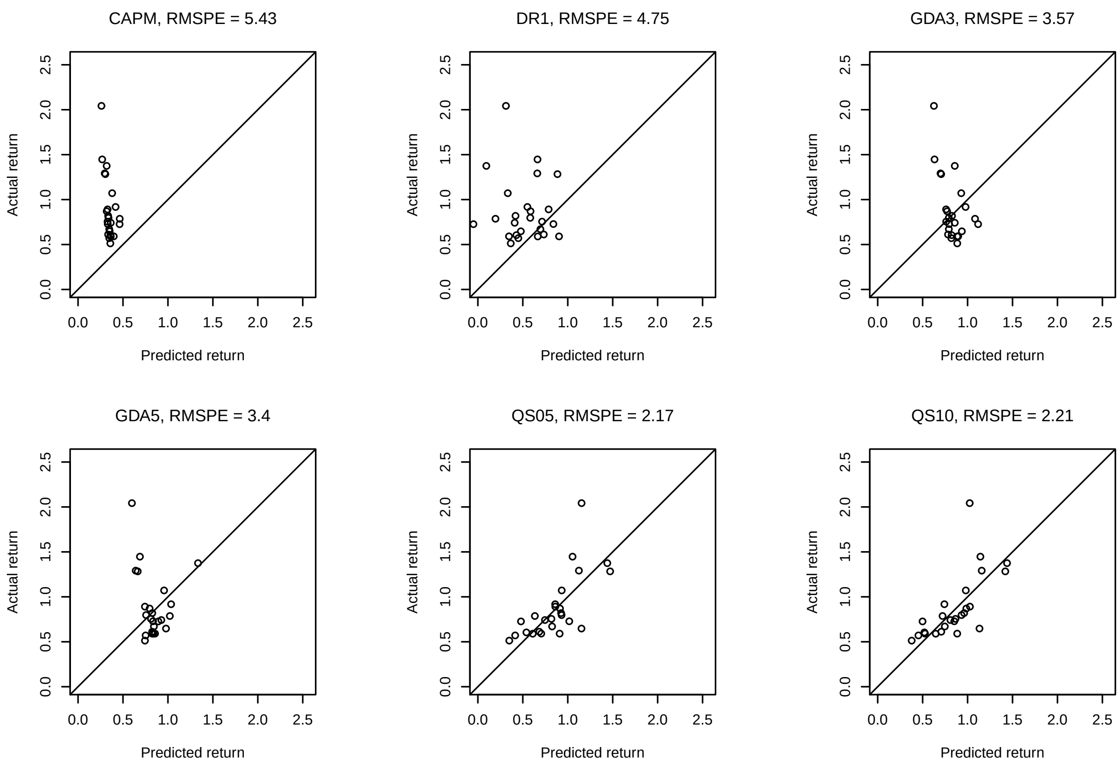

Figure 20: Predicted returns. Plots of predicted versus actual returns for competing models estimated on daily data of 30 Fama-French equal weighted industry portfolios.

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