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The Instrumental Weighted Variables. Part III. Asymptotic Representation

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The Instrumental Weighted Variables. Part III. Asymptotic Representation

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Abstract:
The robust version of the classical instrumental variables, called Instrumental Weighted Variables (IWV) and the conditions for its $\sqrt{n}$-consistency as given in the Part I and II of this paper are recalled. Of course, the reasons why the classical instrumental variables as well as IWV were introduced and the idea of implicit weighting the residuals (firstly employed by the Least Weighted Squares, see Víšek (2000)) are also very briefly recalled (details were discussed in Part I of this paper). Then asymptotic representation and normality of all solutions of the corresponding normal equations is proved.

Keywords: Robustness, instrumental variables, implicit weighting, $\sqrt{n}$-consistency of estimate by instrumental weighted variables, asymptotic representation of the estimate and its normality

AMS classification: 62F35, 62J05

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INTRODUCTION

The paper concludes the proof of basic properties (*Bahadur representation and asymptotic normality*) of *Instrumetal Weighted Variables* (Víšek (2006b) - consistency- and (2006c) - $\sqrt{n}$-consistency). Therefore the reasons for introducing the Instrumental Weighted Variables as well as for employing the idea of implicit weighting residuals, as firstly used in Víšek (2000), are only sketched. First of all, let us introduce basic notations.

Let N denote the set of all positive integers, R the real line and R^p the p-dimensional Euclidean space. We are going to consider the linear regression model given as

$$Y_i = X_i^{\prime}\beta^0 + e_i = \sum_{j=1}^{p} X_{ij}\beta_j^0 + e_i, \quad i = 1, 2, ..., n. \tag{1}$$

To simplify some steps of proofs, we will assume, without loss of generality that $\beta^0 = 0$. Nevertheless sometimes we shall write $\beta - \beta^0$ instead of only β to give e. g. asymptotic representation or the assertion of asymptotic normality of $\sqrt{n}\,(\beta - \beta^0)$ in the usual form. The following conditions are assumed to be fulfilled.

C1 *The sequence* $\left\{(X_i^{\prime}, e_i)^{\prime}\right\}_{i=1}^{\infty}$ *is sequence of independent and identically distributed* $p+1$*-dimensional random vectors (i.i.d. r.v.'s) with absolutely continuous distribution function* $F_{X,e}(x, v)$*. Moreover,* $I\!\!E\left\{(X_1^{\prime}, e)^{\prime} \cdot (X_1^{\prime}, e)\right\}$ *is positive definite matrix and the density* $f_{e|X}(v|X_1 = x)$ *is uniformly in* x *bounded in* v*, say by* U_e*.*

We will use $F_X(x)$ and $F_e(v)$ ($f_X(x)$ and $f_e(v)$) for the marginals of $F_{X,e}(x, v)$ (and their densities, respectively). (Throughout the paper all vectors will be assumed the column ones.)

ESTIMATING BY MEANS OF INSTRUMENTAL VARIABLES

Due to the fact that

$$\hat{\beta}^{(OLS,n)} = \beta^0 + \left(\frac{1}{n}\sum_{k=1}^{n} X_k X_k^{\prime}\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i e_i \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i e_i = I\!\!E\,(X_1 \cdot e_1) \quad \text{a. s.,} \tag{2}$$

one easy verifies that the violation of orthogonality condition $I\!\!E\,\{e_i|X_i\} = 0$ implies inconsistency of the *(Ordinary) Least Squares* (where, of course, due to **C1**, $\frac{1}{n}\sum_{k=1}^{n} X_k X_k^{\prime}$ is, starting with some n_0 (say), positive definite almost surely).

Two of the best known example of the situations when the orthogonality condition fails, were recalled in Víšek (2006b) and (2006c) (the model with lagged explanatory variables and the situation in which the explanatory variables are measured with a random error, see also Judge et al. (1985) or Víšek (1998)).

It is well-known that the classical econometrics offers for such a situation the *Method of Instrumental Variables.*

Definition 1 *For any sequence of random vectors* $\{Z_i\}_{i=1}^{\infty} \subset R^p$ *the solution(s) of the (vector) equation*

$$\sum_{i=1}^{n} Z_i \left(Y_i - X_i^{\prime}\beta\right) = 0 \tag{3}$$

will be called the estimator obtained by means of the method of Instrumental Variables (or Instrumental Variables, for short) and denoted by $\hat{\beta}^{(IV,n)}$.

Nowadays, the method became the standard tool in studies of panel data since the correlation of explanatory variables and disturbances very frequently take place. There is even a collection of papers exploring the optimal way of the selecting the instruments for explanatory variables, see e.g. Arellano, Bond (1991), Arellano, Bover (1995) or Sargan (1988) (and for examples of implementation see for SAS - Der and Everitt (2002), for R and S-PLUS - Fox, J. (2002)).

As (3) is an analogy of the *normal equations* for the *Ordinary Least Squares*, $\hat{\beta}^{(IV,n)}$ is not robust with respect to the outliers and/or leverage points. Hence we are going to define its robustified version. We shall use the idea of *implicit weighting the squared residuals* which was firstly employed in the method of the *Least Weighted Squares*, see Víšek (2000).

RECALLING THE LEAST WEIGHTED SQUARES

Let us enlarge a bit the notations. Let us denote for any $\beta \in R^p$ by $r_i(\beta) = Y_i - X_i'\beta$ the i-th residual and by $r^2_{(h)}(\beta)$ the h-th order statistic among the squared residuals. It means that

$$r^2_{(1)}(\beta) \le r^2_{(2)}(\beta) \le \ \dots \ \le r^2_{(n)}(\beta). \tag{4}$$

Then the *Least Weighted Squares* can be defined as follows (see Víšek (2000), see also (2002a,b)):

$$\hat{\beta}^{(LWS,n,w)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} w_i r^2_{(i)}(\beta) \tag{5}$$

where $w_i, i = 1, 2, ..., n$ are weights (see also Čížek (2002) where the estimator is called the *Smoothed Least Trimmed Squares*). They are usually generated by a weight function with following properties (compare Hájek, Šidák (1967)):

C2 *Weight function* $w : [0,1] \to [0,1]$ *is absolutely continuous and nonincreasing, with the derivative* $w'(\alpha)$ *bounded from below by* $-L$, $w(0) = 1$.

Then put $w_i = w\left(\frac{i-1}{n}\right)$ and (5) turns to

$$\hat{\beta}^{(LWS,n,w)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} w\left(\frac{i-1}{n}\right) r^2_{(i)}(\beta).$$

Following Hájek, Šidák (1967) for any $i \in \{1, 2, ..., n\}$ let us denote by $\pi(\beta, i)$ the rank of the i-th residual. It means that

$$\pi(\beta, i) = j \in \{1, 2, ..., n\} \quad \text{iff} \quad r^2_i(\beta) = r^2_{(j)}(\beta) \tag{6}$$

(notice that $\pi(\beta, i)$ is then r.v.). Then we have

$$\hat{\beta}^{(LWS,n,w)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} w\left(\frac{\pi(\beta,i)-1}{n}\right) r^2_i(\beta). \tag{7}$$

It is straightforward to show that the *Least Weighted Squares* are solution of *normal equations*

$$I\!N E_{X,n}(\beta) = \sum_{i=1}^{n} w\left(\frac{\pi(\beta,i)-1}{n}\right) X_i \left(Y_i - X_i'\beta\right) = 0, \tag{8}$$

see Víšek (2006b).

There is no reason to believe that the inconsistency of the *Ordinary Least Squares*, when the orthogonality condition is broken (as we recalled it in INTRODUCTION), will not take place also for the *Least Weighted Squares*. It is a straightforward idea that the remedy is to "merge" together the *Method of Instrumental Variables* and the *Least Weighted Squares*.

Definition 2 *For any sequence of random vectors* $\{Z_i\}_{i=1}^{\infty} \subset R^p$ *the solution(s) of the (vector) equation*

$$I\!N E_{Z,n}(\beta) = \sum_{i=1}^{n} w\left(\frac{\pi(\beta,i)-1}{n}\right) Z_i \left(Y_i - X_i^{'}\beta\right) = 0 \tag{9}$$

will be called the Instrumental Weighted Variables estimator and denoted by $\hat{\beta}^{(IWV,n,w)}$.

Remark 1 *We of course hope that the Instrumental Weighted Variables are consistent (as the classical Instrumental Variables) and robust (as the Least Weighted Squares). Let us recall that the elements of the sequence* $\{Z_i\}_{i=1}^{\infty}$ *are usually denoted as instruments. Of course, without loss of generality we may assume that* $Z_{i1} = 1$ *and* $I\!E Z_{ij} = 0, j = 2, 3, ..., p$ *and* $i = 1, 2, ...$

We will need some additional notations. For any $\beta \in R^p$ the distribution of the absolute value of residual will be denoted $F_\beta(v)$. In other words,

$$F_\beta(v) = P(|Y_1 - X_1^{'}\beta| < v) = P(|e_1 - X_1^{'}\left(\beta - \beta^0\right)| < v). \tag{10}$$

Similarly, for any $\beta \in R^p$ the empirical distribution of the absolute value of residual will be denoted $F_\beta^{(n)}(v)$. It means that, denoting the indicator of a set A by $I\{A\}$, we have (keep in mind that we have assumed that $\beta^0 = 0$)

$$F_\beta^{(n)}(v) = \frac{1}{n}\sum_{j=1}^{n} I\{|r_j(\beta)| < v\} = \frac{1}{n}\sum_{j=1}^{n} I\left\{|e_j - X_j^{'}\beta| < v\right\}$$

$$= \frac{1}{n}\sum_{j=1}^{n} I\left\{\omega \in \Omega \,:\, |e_j(\omega) - X_j^{'}(\omega)\beta| < v\right\}. \tag{11}$$

It is straightforward that then (for details see Víšek (2006b) or (47) below)

$$F_\beta^{(n)}(|r_i(\beta)|) = \frac{\pi(\beta,i)-1}{n}$$

and so (9) can be written as

$$\sum_{i=1}^{n} w\left(F_\beta^{(n)}(|r_i(\beta)|)\right) Z_i \left(Y_i - X_i^{'}\beta\right) = 0. \tag{12}$$

We will need also the following notations. For any $\beta \in R^p$ the distribution of the product $\beta' Z X' \beta$ will be denoted as $F_{\beta' ZX' \beta}(u)$, i. e.

$$F_{\beta' ZX' \beta}(u) = P(\beta' Z X' \beta < u) \tag{13}$$

and similarly as in previous, the corresponding empirical distribution will be denoted $F^{(n)}_{\beta' ZX' \beta}(u)$, so that

$$F^{(n)}_{\beta' ZX' \beta}(u) = \frac{1}{n} \sum_{j=1}^{n} I\left\{\beta' Z_j X_j' \beta < u\right\}. \tag{14}$$

For any $\zeta \in R^+$ and any $a \in R$ put

$$\gamma_{\zeta,a} = \sup_{\|\beta\|=\zeta} F_{\beta' ZX' \beta}(a). \tag{15}$$

Notice please that due to the fact that the ball $\{\beta \in R^p, \|\beta\| = \zeta\}$ is compact, there is $\beta_\gamma \in \{\beta \in R^p, \|\beta\| = \zeta\}$ so that

$$\gamma_{\zeta,a} = F_{\beta_\gamma' ZX \beta_\gamma}(a). \tag{16}$$

Similarly, for any $\zeta \in R^+$ let us denote

$$\tau_\zeta = - \inf_{\|\beta\| \le \zeta} \beta' I\!E \left[Z_1 X_1' \cdot I\{\beta' Z_1 X_1' \beta < 0\}\right] \beta. \tag{17}$$

Notice please that $\tau_\zeta \ge 0$ and that again due to the fact that the ball $\{\beta \in R^p, \|\beta\| \le \zeta\}$ is also compact, the infimum is finite, and hence there is a $\tilde{\beta} \in \{\beta \in R^p, \|\beta\| \le \zeta\}$ so that

$$\tau_\zeta = -\tilde{\beta}' I\!E \left[Z_1 X_1' \cdot I\{\tilde{\beta}' Z_1 X_1' \tilde{\beta} < 0\}\right] \tilde{\beta}. \tag{18}$$

C3 *The instrumental variables $\{Z_i\}_{i=1}^\infty \subset R^p$ are independent and identically distributed with distribution function $F_Z(z)$. Moreover, they are independent from the sequence $\{e_i\}_{i=1}^\infty$. Further, the joint distribution function $F_{X,Z}(x,z)$ is absolutely continuous, $I\!E\left\{w(F_{\beta^0}(|e_1|)) Z_1 X_1'\right\}$ as well as $I\!E Z_1 Z_1'$ are positive definite and there is $q > 1$ so that $I\!E\left\{\|Z_1\| \cdot \|X_1\|\right\}^q < \infty$. Finally, there is $a > 0$, $b \in (0,1)$ and $\lambda > 0$ so that*

$$a \cdot (b - \gamma_{\lambda,a}) \cdot w(b) > \tau_\lambda \tag{19}$$

for $\gamma_{\lambda,a}$ and τ_λ given by (15) and (17).

Remark 2 *Please, compare* **C3** *with Víšek (1998a,b) where we considered instrumental M-estimators and the discussion of assumptions for M-instrumental variables was given.*

C4 *The vector equation*

$$\beta' I\!E \left[w\left(F_\beta(|r_1(\beta)|)\right) Z_1 \left(e_1 - X_1'\beta\right)\right] = 0 \tag{20}$$

in the variable $\beta \in R^p$ has unique solution $\beta^0 = 0$.

Lemma 1 *Let the conditions* **C1**, **C2**, **C3** *and* **C4** *be fulfilled. Then any sequence* $\left\{\hat{\beta}^{(IWV,n,w)}\right\}_{n=1}^{\infty}$ *of the solutions of normal equations* $I\!N E_{Z,n}(\hat{\beta}^{(IWV,n,w)}) = 0$ *(see (9)) is weakly consistent.*

For the proof see Víšek (2006b).

$\sqrt{n}$-CONSISTENCY OF THE INSTRUMENTAL WEIGHTED VARIABLES

We will need to enlarge the previous conditions.

NC1 *The density* $f_{e|X}(r|X_1 = x)$ *is uniformly with respect to* x *Lipschitz of the first order (with the corresponding constant equal to* B_e*). Moreover,* $f_e'(r)$ *exists and is bounded in absolute value by* U_e'*.*

NC2 *The derivative* $w'(\alpha)$ *of the weight function is Lipschitz of the first order (with the corresponding constant* J_w*).*

Lemma 2 *Let the conditions* **C1**, **C2**, **C3**, **C4**, **NC1** *and* **NC2** *be fulfilled. Then any sequence* $\left\{\hat{\beta}^{(IWV,n,w)}\right\}_{n=1}^{\infty}$ *of the solutions of normal equations* $I\!N E_{Z,n}(\hat{\beta}^{(IWV,n,w)}) = 0$ *is* $\sqrt{n}$*-consistent, i.e.*

$$\forall(\varepsilon > 0) \quad \exists(K < \infty, n_\varepsilon \in N) \quad \forall(n > n_\varepsilon) \quad : \qquad P\left(\sqrt{n}\left(\hat{\beta}^{(IWV,n,w)} - \beta^0\right) > K\right) < \varepsilon.$$

For the proof see Víšek (2006c).

ASYMPTOTIC REPRESENTATION OF THE INSTRUMENTAL WEIGHTED VARIABLES

First of all, let us put for any $M \in R^+$

$$\mathcal{T}(M) = \{t \in R^p, \ \|t\| \le M\}. \tag{21}$$

Then we can prove

Lemma 3 *Let the conditions* **C1** *and* **NC1** *hold and fix arbitrary* $\varepsilon > 0$, $M \in (0, \infty)$ *and* $\tau \in (\frac{1}{2}, \frac{3}{4})$*. Then there is* $K \in (0, \infty)$ *and* $n_{\varepsilon,M,\tau} \in N$ *so that for all* $n > n_{\varepsilon,M,\tau}$

$$P\left(\left\{\omega \in \Omega : \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} n^\tau \left| F^{(n)}_{\beta^0 - n^{-\frac{1}{2}}t}(r) - F^{(n)}_{\beta^0}(r)\right| < K\right\}\right) > 1 - \varepsilon. \tag{22}$$

Remark 3 *Let us recall that* $F^{(n)}_{\beta^0 - n^{-\frac{1}{2}}t}$ *and* $F^{(n)}_{\beta^0}$ *are random variables which we could indicate by writing (22) in the following form*

$$P\left(\left\{\omega \in \Omega : \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} n^\tau \left| F^{(n)}_{\beta^0 - n^{-\frac{1}{2}}t}(r, \omega) - F^{(n)}_{\beta^0}(r, \omega)\right| < K\right\}\right) > 1 - \varepsilon.$$

Proof of Lemma 3: Since both d. f. $F^{(n)}_{\beta^0-n^{-\frac{1}{2}}t}(r)$ and $F^{(n)}_{\beta^0}(r)$ are equal to zero for $r \leq 0$, we shall consider only $r > 0$. Taking into account that we have assumed that $\beta^0 = 0$, we have (see (11))

$$F^{(n)}_{\beta^0}(r) = \frac{1}{n}\sum_{i=1}^{n} I\left\{|e_i| < r\right\} = \frac{1}{n}\sum_{i=1}^{n} I\left\{\omega \in \Omega \ : \ |e_i(\omega)| < r\right\}$$

and

$$F^{(n)}_{\beta^0-n^{-\frac{1}{2}}t}(r) = \frac{1}{n}\sum_{i=1}^{n} I\left\{|e_i + n^{-\frac{1}{2}}X_i't| < r\right\} = \frac{1}{n}\sum_{i=1}^{n} I\left\{\omega \in \Omega \ : \ |e_i(\omega) + n^{-\frac{1}{2}}X_i'(\omega)t| < r\right\}.$$

Denote by $\#A$ the number of elements of the set A and put for any $r \in R^+$ and $t \in M$

$$m^{(+)}_{n,U}(r,t) = \#\left\{i \in \{1,2,...,n\} \ : \ e_i \geq r \ \ and \ \ \left|e_i + n^{-\frac{1}{2}}X_i't\right| < r\right\}, \tag{23}$$

$$m^{(-)}_{n,U}(r,t) = \#\left\{i \in \{1,2,...,n\} \ : \ |e_i| < r \ \ and \ \ e_i + n^{-\frac{1}{2}}X_i't \geq r\right\}, \tag{24}$$

$$m^{(+)}_{n,L}(r,t) = \#\left\{i \in \{1,2,...,n\} \ : \ e_i \leq -r \ \ and \ \ \left|e_i + n^{-\frac{1}{2}}X_i't\right| < r\right\}, \tag{25}$$

$$m^{(-)}_{n,L}(r,t) = \#\left\{i \in \{1,2,...,n\} \ : \ |e_i| < r \ \ and \ \ e_i + n^{-\frac{1}{2}}X_i't \leq -r\right\} \tag{26}$$

and

$$m_n(r,t) = m^{(+)}_{n,U}(r,t) - m^{(-)}_{n,U}(r,t) + m^{(+)}_{n,L}(r,t) - m^{(-)}_{n,L}(r,t).$$

Now, let us observe that $m^{(+)}_{n,U}(r,t)$ represents the number of indices for which

$$\left|e_i + n^{-\frac{1}{2}}X_i't\right| < r \tag{27}$$

but for which

$$r \leq e_i. \tag{28}$$

In other words, the observations, indices of which belong to $m^{(+)}_{n,U}(r,t)$, are taken into account when $F^{(n)}_{\beta^0-n^{-\frac{1}{2}}t}(r)$ is evaluated but they are not assumed when we look for $F^{(n)}_{\beta^0}(r)$. Of course, both for $F^{(n)}_{\beta^0}(r)$ as well as for $F^{(n)}_{\beta^0-n^{-\frac{1}{2}}t}(r)$ there are some other indices which are taken into account. They appear in other sets given in (24), (25) and (26). It means that similar conclusions are valid for

$$m^{(-)}_{n,U}(r,t), \quad m^{(+)}_{n,L}(r,t) \quad \text{and} \quad m^{(-)}_{n,L}(r,t)$$

(the subindices U and L indicate that we take into account upper and lower end of the interval $(-r,r)$, respectively; similarly superindices "(+)" and "(-)" hint that the corresponding number $m^{(*)}_{*,*}$ is add to or subtracted from the number of points taken into account for evaluating the e.d.f. when "moving" from β^0 to $\beta^0 - n^{-\frac{1}{2}}t$). Hence

$$\left|F^{(n)}_{\beta^0-n^{-\frac{1}{2}}t}(r) - F^{(n)}_{\beta^0}(r)\right| \leq \frac{1}{n}\left|m_n(r,t)\right|.$$

It means that if we prove that for any $\varepsilon \in (0,1)$, $M > 0$ and $\tau \in (\frac{1}{2},\frac{3}{4})$ there is $n_{\varepsilon,M,\tau} \in N$ so that for all $n > n_{\varepsilon,M,\tau}$

$$P\left(\left\{\omega \in \Omega : n^{\tau-1}\sup_{r\in R}\ \sup_{t\in\mathcal{T}(M)}|m_n(r,t)| < K\right\}\right) > 1-\varepsilon,$$

we conclude the proof. In order to do it, let us consider (27) and (28) which implies that

$$r \leq e_i < r - n^{-\frac{1}{2}} X_i' t$$

under assumption that $r < r - n^{-\frac{1}{2}} X_i' t$. Otherwise, there is no possibility to fulfill (27) and (28) simultaneously. So, putting

$$b_i^{(+)}(r,t) = I\left\{r \leq e_i < r - n^{-\frac{1}{2}} X_i' t\right\}, \tag{29}$$

we have

$$m_{n,U}^{(+)}(r,t) \leq \sum_{i=1}^{n} b_i^{(+)}(r,t). \tag{30}$$

Further put

$$\xi_i^{(+)}(r,t) = b_i^{(+)}(r,t) - \mathbb{E} b_i^{(+)}(r,t) \tag{31}$$

and denote

$$\pi_i(r,t) = \mathbb{E} b_i^{(+)}(r,t).$$

Notice please that $\left\{\xi_i^{(+)}(r,t)\right\}_{i=1}^{\infty}$ is a sequence of independent identically distributed stochastic processes with index set $R \times \mathcal{T}(M)$ (i.e. $r \in R, t \in \mathcal{T}(M)$ (for $\mathcal{T}(M)$ see (21)). Each of this processes is separable (the definition of *separability* is recalled in Appendix). So, we have (for U_e see **C1**)

$$\pi_i(r,t) = \int I\left\{r \leq v < r n^{-\frac{1}{2}} x' t\right\} \mathrm{d}F_{X,e}(x,v)$$

$$= \int \left[\int I\left\{r \leq v < r - n^{-\frac{1}{2}} x' t\right\} f_{e|X}(v|X=x)\mathrm{d}v\right] \mathrm{d}F_X(x)$$

$$\leq n^{-\frac{1}{2}} \cdot U_e \|t\| \int \|x\| \,\mathrm{d}F_X(x) \leq n^{-\frac{1}{2}} \cdot U_e \cdot M \cdot \mathbb{E}\|X_1\|. \tag{32}$$

Now denote $\Delta = U_e \cdot M \cdot \mathbb{E}\|X_1\|$ and find $n_0 \in N$ so that for all $n > n_0$ we have $n^{-\frac{1}{2}} \cdot \Delta \in (0,1)$. It means that on $\mathcal{T}(M)$ we have for $n > n_0$

$$\pi_i(r,t) < n^{-\frac{1}{2}} \cdot \Delta. \tag{33}$$

In what follows consider only $n > n_0$. We have

$$P\left(\xi_i^{(+)}(r,t) = 1 - \pi_i(r,t)\right) = \pi_i(r,t)$$

and

$$P\left(\xi_i^{(+)}(r,t) = -\pi_i(r,t)\right) = 1 - \pi_i(r,t).$$

Now, following Portnoy (1983), Jurečková (1984) or Jurečková and Sen (1989), we are going to employ Lemma A.1. Let us recall that due to the definition of $\xi_i^{(+)}(r,t)$, $\left\{\xi_i^{(+)}(r,t)\right\}_{i=1}^{\infty}$ is a sequence of i.i.d. r.v.'s. Let us denote by $W(s)$ the Wiener process and let us define $\tau_i^{(+)}(r,t)$ to be the time for the Wiener process to exit interval $(-\pi_i(r,t), 1 - \pi_i(r,t))$. Then $\xi_i^{(+)}(r,t) =_{\mathcal{D}} W(\tau_i^{(+)}(r,t))$ and hence

$$n^{-\frac{1}{4}} \sum_{i=1}^{n} \xi_i^{(+)}(r,t) =_{\mathcal{D}} n^{-\frac{1}{4}} \sum_{i=1}^{n} W(\tau_i^{(+)}(r,t)) =_{\mathcal{D}} W\left(n^{-\frac{1}{2}} \sum_{i=1}^{n} \tau_i^{(+)}(r,t)\right)$$

(for details of this step see again Portnoy (1983), Jurečková (1984), Jurečková and Sen (1989) or Víšek (1996a), (2002c)). Further, let us define V_i to be the time for the Wiener process to exit interval $(-n^{-\frac{1}{2}} \cdot \Delta, 1)$. Due to the fact that for any $r \in R$ and any $t \in \mathcal{T}(M)$ we have for all $i = 1, 2, ..., n$

$$\pi_i(r,t)) \le n^{-\frac{1}{2}} \cdot \Delta \quad \text{and} \quad 1 - \pi_i(r,t)) \le 1,$$

we conclude that again for any $r \in R^+$ and any $t \in \mathcal{T}(M)$

$$\tau_i^{(+)}(r,t) < V_i.$$

Then, employing separability of the processes $\xi_i^{(+)}(r,t), \quad i = 1, 2, ...$ and applying Lemma A.6, we arrive at

$$n^{-\frac{1}{4}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} \left| \sum_{i=1}^{n} \xi_i^{(+)}(r,t) \right| =_{\mathcal{D}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} \left| W \left(n^{-\frac{1}{2}} \sum_{i=1}^{n} \tau_i^{(+)}(r,t) \right) \right|$$
$$\le \sup_{s \in \left(0, n^{-\frac{1}{2}} \sum_{i=1}^{n} V_i\right)} |W(s)|. \tag{34}$$

Due to Lemma A.1 we have

$$n^{-\frac{1}{2}} \cdot I\!\!E V_i = n^{-1} \cdot \Delta.$$

Finding $K_1 < \infty$ so that $\frac{\Delta}{K_1} < \frac{\varepsilon}{2}$ and utilizing Chebyshev inequality for positive random variables, we have for all $n \in N$

$$P\left(\left\{\omega \in \Omega \;:\; n^{-\frac{1}{2}} \sum_{i=1}^{n} V_i > K_1 \right\}\right) \le \frac{\Delta}{K_1} < \frac{\varepsilon}{2}. \tag{35}$$

Now, we can find $K_2 > 0$ so that $\frac{4 \cdot K_1}{K_2^2} \le \frac{\varepsilon}{2}$ and utilizing Lemma A.2, (34) and (35), we obtain

$$P\left(n^{-\frac{1}{4}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} \left| \sum_{i=1}^{n} \xi_i^{(+)}(r,t) \right| > K_2 \right)$$

$$\le P\left(\sup_{s \in \left(0, n^{-\frac{1}{2}} \sum_{i=1}^{n} V_i\right)} |W(s)| > K_2 \right)$$

$$= P\left(\left\{ \sup_{s \in \left(0, n^{-\frac{1}{2}} \sum_{i=1}^{n} V_i\right)} |W(s)| > K_2 \right\} \cap \left\{ n^{-\frac{1}{2}} \sum_{i=1}^{n} V_i \le K_1 \right\} \right)$$

$$+ P\left(\left\{ \sup_{s \in \left(0, n^{-\frac{1}{2}} \sum_{i=1}^{n} V_i\right)} |W(s)| > K_2 \right\} \cap \left\{ n^{-\frac{1}{2}} \sum_{i=1}^{n} V_i > K_1 \right\} \right)$$

$$\le P\left(\sup_{0 \le s \le K_1} |W(s)| > K_2 \right) + \frac{\varepsilon}{2} \le 2 \cdot P\left(|W(2 \cdot K_1)| > K_2\right) + \frac{\varepsilon}{2}. \tag{36}$$

Now, recalling the fact that var $\{W(2 \cdot K_1)\} = 2 \cdot K_1$ and using Chebyshev inequality, we arrive at

$$2 \cdot P\left(|W(2 \cdot K_1)| > K_2\right) \leq \frac{4 \cdot K_1}{K_2^2} \leq \frac{\varepsilon}{2}. \tag{37}$$

Finally, (35), (36) together with (37) imply

$$P\left(n^{-\frac{1}{4}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} \left|\sum_{i=1}^{n} \xi_i^{(+)}(r,t)\right| > K_2\right) \leq \varepsilon. \tag{38}$$

Taking now successively into account (24), (25) and (26), we define

$$b_i^{(-)}(r,t) = I\left\{r - n^{-\frac{1}{2}} X_i' t \leq e_i < r\right\}, \qquad c_i^{(+)}(r,t) = I\left\{-r - n^{-\frac{1}{2}} X_i' t < e_i \leq -r\right\},$$

$$c_i^{(-)}(r,t) = I\left\{-r < e_i \leq -r - n^{-\frac{1}{2}} X_i' t\right\}$$

and we have

$$n^{-\frac{1}{4}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} |m_{n,U}(r,t)| \leq n^{-\frac{1}{4}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} \left|\sum_{i=1}^{n} \left[b_i^{(+)}(r,t) - b_i^{(-)}(r,t) + c_i^{(+)}(r,t) - c_i^{(-)}(r,t)\right]\right|$$

$$\leq n^{-\frac{1}{4}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} \left|\sum_{i=1}^{n} \left[\left(b_i^{(+)}(r,t) - I\!E b_i^{(+)}(r,t)\right) - \left(b_i^{(-)}(r,t) - I\!E b_i^{(-)}(r,t)\right)\right.\right.$$

$$\left.\left. + \left(c_i^{(+)}(r,t) - I\!E c_i^{(+)}(r,t)\right) - \left(c_i^{(-)}(r,t) - I\!E c_i^{(-)}(r,t)\right)\right]\right| \tag{39}$$

$$+ n^{-\frac{1}{4}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} \left|\sum_{i=1}^{n} \left[I\!E b_i^{(+)}(r,t) - I\!E b_i^{(-)}(r,t) + I\!E c_i^{(+)}(r,t) - I\!E c_i^{(-)}(r,t)\right]\right|. \tag{40}$$

Having derived successively similar inequalities to (38) for

$$\xi_i^{(-)}(r,t) = b_i^{(-)}(r,t) - I\!E b_i^{(-)}(r,t), \qquad \zeta_i^{(+)}(r,t) = c_i^{(+)}(r,t) - I\!E c_i^{(+)}(r,t)$$

and

$$\zeta_i^{(-)}(r,t) = c_i^{(-)}(r,t) - I\!E c_i^{(-)}(r,t),$$

we conclude that (39) is bounded in probability. Now, to be able to make the same conclusion about (40), we need to estimate the corresponding mean values. Due to the fact that $b_i^{(+)}(r,t) \geq 0, b_i^{(-)}(r,t) \geq 0, c_i^{(+)}(r,t) \geq 0$ and $c_i^{(-)}(r,t) \geq 0$, all mean values are nonnegative. Returning to (29), we may write

$$I\!E b_i^{(+)}(r,t) = \int I\left\{r \leq v < r - n^{-\frac{1}{2}} x' t\right\} \mathrm{d}F_{X,e}(x,v)$$

$$= \int \left[\int I\left\{r \leq v < r - n^{-\frac{1}{2}} x' t\right\} f_{e|X}(v|X=x)\mathrm{d}v\right] \mathrm{d}F_X(x)$$

$$= \int \left[\int_r^{r - n^{-\frac{1}{2}} x' t} f_{e|X}(v|X=x)\mathrm{d}v\right] \mathrm{d}F_X(x)$$

$$= \int \left[\int_r^{r-n^{-\frac{1}{2}}x't} \left(f_{e|X}(v|X=x) - f_{e|X}(r|X=x) \right) \mathrm{d}v \right] \mathrm{d}F_X(x)$$

$$+ \int \left[f_{e|X}(r|X=x) \int_r^{r-n^{-\frac{1}{2}}x't} \mathrm{d}v \right] \mathrm{d}F_X(x).$$

So

$$I\!\!E b_i^{(+)}(r,t) = n^{-\frac{1}{2}} \int |x't| \, f_{e|X}(r|X=x) \mathrm{d}F_X(x) + R_b^{(+)}(r,t)$$

where

$$\left| R_b^{(+)}(r,t) \right| \leq \frac{1}{n} B_e \int (x't)^2 \, \mathrm{d}F_X(x).$$

The analogous expressions we obtain for $I\!\!E b_i^{(-)}(r,t)$, $I\!\!E c_i^{(+)}(r,t)$ and $I\!\!E c_i^{(-)}(r,t)$, e. g.

$$I\!\!E c_i^{(-)}(r,t) = n^{-\frac{1}{2}} \int |x't| \, f_{e|X}(r|X=x) \mathrm{d}F_X(x) + R_c^{(-)}(r,t)$$

(notice please that the first term of the right hand side is the same for all mean values $I\!\!E b_i^{(+)}(r,t)$, $I\!\!E b_i^{(-)}(r,t)$, $I\!\!E c_i^{(+)}(r,t)$ and $I\!\!E c_i^{(-)}(r,t)$) where again

$$\left| R_c^{(-)}(r,t) \right| \leq \frac{1}{n} B_e \int (x't)^2 \, \mathrm{d}F_X(x).$$

Hence

$$n^{-\frac{1}{4}} \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} \left| \sum_{i=1}^n \left[I\!\!E b_i^{(+)}(r,t) - I\!\!E b_i^{(-)}(r,t) + I\!\!E c_i^{(+)}(r,t) - I\!\!E c_i^{(-)}(r,t) \right] \right|$$

$$\leq 4 \cdot n^{-\frac{1}{4}} \sup_{t \in \mathcal{T}(M)} B_e \cdot \int (x't)^2 \, \mathrm{d}F_X(x)$$

$$\leq 4 \cdot n^{-\frac{1}{4}} B_e \cdot M^2 \int \|x\|^2 \, \mathrm{d}F_X(x) = O(n^{-\frac{1}{4}}).$$

Since $n^{\tau-1} < n^{-\frac{1}{4}}$, we conclude the proof. □

Corollary 1 *Let the conditions* **C1**, **C2**, **C3**, **C4**, **NC1** *and* **NC2** *hold and fix arbitrary* $\varepsilon > 0$, $M \in (0,\infty)$ *and* $\tau \in (\frac{1}{2}, \frac{3}{4})$. *Then there is* $K \in (0,\infty)$ *and* $n_{\varepsilon,M,\tau} \in N$ *so that for all* $n > n_{\varepsilon,M,\tau}$

$$P\left(\left\{ \omega \in \Omega : \sup_{r \in R} n^\tau \left| F^{(n)}_{\hat{\beta}(IWV,n,w)}(r) - F^{(n)}_{\beta^0}(r) \right| < K \right\} \right) > 1 - \varepsilon.$$

Proof follows immediately from Lemmas 2 and 3. □

Lemma 4 *Let the conditions* **C1** *and* **NC1** *hold and fix arbitrary* $\varepsilon > 0$, $M \in (0,\infty)$ *and* $\tau \in (\frac{1}{2}, \frac{3}{4})$. *Then there is* $K \in (0,\infty)$ *and* $n_{\varepsilon,M,\tau} \in N$ *so that for all* $n > n_{\varepsilon,M,\tau}$

$$P\left(\left\{ \omega \in \Omega : \sup_{r \in R} \sup_{t \in \mathcal{T}(M)} n^\tau \left| F^{(n)}_{\beta^0}\left(\left| r - n^{-\frac{1}{2}} X_1' t \right| \right) - F^{(n)}_{\beta^0}(r) \right| < K \right\} \right) > 1 - \varepsilon. \quad (41)$$

Proof: Let $r > 0$. Recalling that

$$F^{(n)}_{\beta^0}\left(\left|r - n^{-\frac{1}{2}}X_1't\right|\right) - F^{(n)}_{\beta^0}(r) = \frac{1}{n}\sum_{i=1}^{n}\left[I\left\{|e_i| < \left|r - n^{-\frac{1}{2}}X_1't\right|\right\} - I\left\{|e_i| < r\right\}\right],$$

we can nearly repeat the proof of Lemma 3. □

Corollary 2 *Let the conditions* **C1**, **C2**, **C3**, **C4**, **NC1** *and* **NC2** *hold and fix arbitrary* $\varepsilon > 0$, $M \in (0,\infty)$ *and* $\tau \in (\frac{1}{2}, \frac{3}{4})$. *Then there is* $K \in (0,\infty)$ *and* $n_{\varepsilon,M,\tau} \in N$ *so that for all* $n > n_{\varepsilon,M,\tau}$

$$P\left(\left\{\omega \in \Omega : \max_{\{i=1,2,\dots,n\}} n^{\tau}\left|F^{(n)}_{\beta^0}\left(\left|r_i(\hat{\beta}^{(IWV,n,w)})\right|\right) - F^{(n)}_{\beta^0}(|e_i|)\right| < K\right\}\right) > 1 - \varepsilon.$$

Proof follows immediately from Lemmas 2 and 4. □

Returning to (12), let us recall that (due to the fact that we have assumed $\beta^0 = 0$)

$$\sum_{i=1}^{n} w\left(F^{(n)}_{\hat{\beta}^{(IWV,n,w)}}(|r_i(\hat{\beta}^{(IWV,n,w)})|)\right) Z_i\left(Y_i - X_i'\hat{\beta}^{(IWV,n,w)}\right)$$

$$= \sum_{i=1}^{n} w\left(F^{(n)}_{\hat{\beta}^{(IWV,n,w)}}(|r_i(\hat{\beta}^{(IWV,n,w)})|)\right) Z_i\left(e_i - X_i'\hat{\beta}^{(IWV,n,w)}\right) = 0. \tag{42}$$

Lemma 5 *Let the conditions* **C1**, **C2**, **C3**, **C4**, **NC1** *and* **NC2** *hold. Then*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w\left(F^{(n)}_{\beta^0}(|e_i|)\right) Z_i\left(Y_i - X_i'\hat{\beta}^{(IWV,n,w)}\right) = o_p(1). \tag{43}$$

Proof: Let us recall that we have assumed $\beta^0 = 0$ and write

$$\frac{1}{\sqrt{n}}\left\|\sum_{i=1}^{n} w\left(F^{(n)}_{\beta^0}(|e_i|)\right) Z_i\left(Y_i - X_i'\hat{\beta}^{(IWV,n,w)}\right)\right\|$$

$$= \frac{1}{\sqrt{n}}\left\|\sum_{i=1}^{n} w\left(F^{(n)}_{\beta^0}(|e_i|)\right) Z_i\left(e_i - X_i'\hat{\beta}^{(IWV,n,w)}\right)\right\|$$

$$\leq \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left|w\left(F^{(n)}_{\beta^0}(|e_i|)\right) - w\left(F^{(n)}_{\beta^0}(|r_i(\hat{\beta}^{(IWV,n,w)})|)\right)\right| \cdot \|Z_i\| \cdot \left|e_i - X_i'\hat{\beta}^{(IWV,n,w)}\right| \tag{44}$$

$$+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left|w\left(F^{(n)}_{\beta^0}(|r_i(\hat{\beta}^{(IWV,n,w)})|)\right) - w\left(F^{(n)}_{\hat{\beta}^{(IWV,n,w)}}(|r_i(\hat{\beta}^{(IWV,n,w)})|)\right)\right|$$

$$\times \|Z_i\| \cdot \left|e_i - X_i'\hat{\beta}^{(IWV,n,w)}\right| \tag{45}$$

$$+\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w\left(F^{(n)}_{\hat{\beta}^{(IWV,n,w)}}(|r_i(\hat{\beta}^{(IWV,n,w)})|)\right) Z_i\left(e_i - X_i'\hat{\beta}^{(IWV,n,w)}\right)\right|. \tag{46}$$

First of all, notice that (46) is equal to zero, see (42). Now, let us consider (44). It is not larger than

$$\frac{1}{n^{\tau+\frac{1}{2}}} \cdot n^{\tau} \sup_{r\in R} \left| w\left(F_{\beta^0}^{(n)}(|r|)\right) - w\left(F_{\hat{\beta}^{(IWV,n,w)}}^{(n)}(|r|)\right)\right| \cdot \sum_{i=1}^{n} \|Z_i\| \cdot |e_i|$$

$$+\frac{1}{n^{\tau+\frac{1}{2}}} \cdot n^{\tau} \sup_{r\in R} \left| w\left(F_{\beta^0}^{(n)}(|r|)\right) - w\left(F_{\hat{\beta}^{(IWV,n,w)}}^{(n)}(|r|)\right)\right| \cdot \sum_{i=1}^{n} \|Z_i\| \cdot \|X_i\| \cdot \left\|\hat{\beta}^{(IWV,n,w)} - \beta^0\right\|.$$

Since $\tau + \frac{1}{2} > 1$, $I\!E \, \|Z_i\| < \infty$, $I\!E|e_i| < \infty$ etc., due to Corollary 2 we conclude that (44) is $o_p(1)$. Along similar lines, employing Corollary 1, we find that also (45) is $o_p(1)$, and due to (42) the proof follows. □

Recalling that

$$F_{\beta}^{(n)}(v) = \frac{1}{n}\sum_{j=1}^{n} I\left\{|r_j(\beta)| < v\right\}$$

and that $r_j(\beta^0) = e_j$, we find that

$$F_{\beta^0}^{(n)}(|e_i|) = \frac{1}{n}\sum_{j=1}^{n} I\{|e_j| < |e_i|\}.$$

But $\sum_{j=1}^{n} I\{|e_j| < |e_i|\}$ represents number of indices for which absolute value of disturbance e_j is smaller that $|e_i|$, i. e. $\pi(\beta^0, i) - 1$, see (6). Finally,

$$F_{\beta^0}^{(n)}(|e_i|) = \frac{\pi(\beta^0, i) - 1}{n} \tag{47}$$

(see also Víšek (2006b), (15)). Denoting further

$$w_k^* = w\left(\frac{k-1}{n}\right) - w\left(\frac{k}{n}\right),$$

we have

$$w\left(\frac{k-1}{n}\right) = \sum_{j=k}^{n} w_j^*.$$

Now, recalling that $\pi(\beta, i) = j$ iff $r_i^2(\beta) = r_{(j)}^2(\beta)$ (i. e. that $r_i^2(\beta) = r_{(\pi(\beta,i))}^2$), we can easy verify that

$$w\left(F_{\beta^0}^{(n)}(|e_i|)\right) = w\left(\frac{\pi(\beta^0, i) - 1}{n}\right) = \sum_{\ell=\pi(\beta^0,i)}^{n} w_\ell^* = \sum_{\ell=1}^{n} w_\ell^* \cdot I\{r_i^2(\beta^0) \le r_{(\ell)}^2(\beta^0)\} \tag{48}$$

and hence (43) can be rewritten as follows (let's recall that $r_i^2(\beta^0) = e_i^2$)

$$\frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_i \left(Y_i - X_i'\hat{\beta}^{(IWV,n,w)}\right) \cdot I\{e_i^2 \le e_{(\ell)}^2\} = o_p(1). \tag{49}$$

Now, let $G(z)$ denote the distribution function of e_1^2 and let for any $\alpha \in (0, 1)$, u_α^2 stay for the upper α-quantile of $G(z)$, i. e. $P(e_1^2 > u_\alpha^2) = 1 - G(u_\alpha^2) = \alpha$. Moreover, since $G(z) = F(\sqrt{z}) - F(-\sqrt{z})$, there is a density of $G(z)$, say $g(z)$.

Lemma 6 *Let $\{e_i\}_{i=1}^{\infty}$ ($e_i \in R$) be a sequence of independent and identically distributed random variables with absolutely continuous distribution function $F_e(r)$. For any $\alpha \in (0,1)$ let u_α^2 be the upper quantile of the distribution $G(z)$ and for any $n \in N$ put $\ell_n(\alpha) = [(1-\alpha)n]_{int}$. Then for any $\varepsilon \in (0,1)$ there is a finite constant $K^{(\varepsilon)}$ and $n_\varepsilon \in N$ such that for all $n > n_\varepsilon$ and any $\alpha \in (0,1)$ there is an interval $I_{\alpha,n}^{(\varepsilon)}$ such that*

$$u_\alpha^2 \in I_{\alpha,n}^{(\varepsilon)} \quad for\ all \quad \alpha \in (0,1), \tag{50}$$

$$P\left(\bigcap_{\alpha\in(0,1)} \left\{\omega \in \Omega \ :\ r^2_{(\ell_n(\alpha))}(\beta^0) \in I_{\alpha,n}^{(\varepsilon)}\right\}\right) > 1-\varepsilon, \tag{51}$$

$$\sup_{\alpha\in(0,1)} P\left(e_i^2 \in I_{\alpha,n}^{(\varepsilon)}\right) \le n^{-\frac{1}{2}} K^{(\varepsilon)} \tag{52}$$

and

$$\sup_{\alpha\in(0,1)} I\!E\left[|e_i| \cdot I_{\alpha,n}^{(\varepsilon)}\right] \le n^{-\frac{1}{2}} K^{(\varepsilon)}. \tag{53}$$

Proof: During this proof let us write briefly ℓ_n instead of $\ell_n(\alpha)$. At first, we shall show that for any $\varepsilon > 0$ there is $K_\varepsilon < \infty$ and $n_\varepsilon \in N$ such that for all $n > n_\varepsilon$ and any $\alpha \in (0,1)$ there is a $U_{\alpha,n}^{(\varepsilon)}$ (which can be infinite) such that

$$u_\alpha^2 \le U_{\alpha,n}^{(\varepsilon)} \quad \text{for all} \quad \alpha \in (0,1), \tag{54}$$

$$P\left(\bigcap_{\alpha\in(0,1)} \left\{\omega \in \Omega \ :\ r^2_{(\ell_n)}(\beta^0) \le U_{\alpha,n}^{(\varepsilon)}\right\}\right) > 1-\tfrac{1}{2}\varepsilon, \tag{55}$$

$$\sup_{\alpha\in(0,1)} P\left(e_i^2 \in \left[u_\alpha^2, U_{\alpha,n}^{(\varepsilon)}\right]\right) < \tfrac{1}{2} n^{-\frac{1}{2}} K_\varepsilon. \tag{56}$$

and

$$\sup_{\alpha\in(0,1)} I\!E\left[|e_i|\ I\{u_\alpha^2 \le |e_i| \le U_{\alpha,n}^{(\varepsilon)}\}\right] = \sup_{\alpha\in(0,1)} \int_{z^2\in\left[u_\alpha^2, U_{\alpha,n}^{(\varepsilon)}\right]} |z| f_e(z) \mathrm{d}z < \tfrac{1}{2} n^{-\frac{1}{2}} K_\varepsilon. \tag{57}$$

Let us fix an $\varepsilon > 0$ and denote $W(s)$ the Wiener process. Further, employing Lemma A.2, let us find $K_\varepsilon < \infty$ such that

$$P\left(\sup_{0\le s\le 2} |W(s)| > \tfrac{1}{2}K_\varepsilon\right) < \tfrac{1}{4}\varepsilon. \tag{58}$$

Further, let us define a sequence of i.i.d. r.v. $\{V_i\}_{i=1}^{\infty}$

$$V_i = \text{time for } W(s) \text{ to exit the interval } (-1,1)$$

and applying Lemma A.1, we have $I\!EV_i = 1$ for all i. Now, for $\varepsilon > 0$ (we have fix
ed a few lines above) let us find $n_\varepsilon \in N$ such that for any $n > n_\varepsilon$, putting

$$B_n = \left\{\omega \in \Omega \ :\ n^{-1}\sum_{i=1}^{n} V_i > 2\right\},$$

we have

$$P(B_n) < \tfrac{1}{4}\varepsilon. \tag{59}$$

In what follows let us assume only $n > n_\varepsilon$. Finally, let $U^{(\varepsilon)}_{\alpha,n}$ be such that

$$\max\left\{\int_{z^2\in\left[u^2_\alpha,U^{(\varepsilon)}_{\alpha,n}\right]}|z|f_e(z)\mathrm{d}z, G\left(U^{(\varepsilon)}_{\alpha,n}\right)-G\left(u^2_\alpha\right)\right\} = \min\left\{n^{-\frac{1}{2}}K_\varepsilon, \int_{z^2\in[u^2_\alpha,\infty)}|z|f_e(z)\mathrm{d}z, \alpha\right\}. \tag{60}$$

It is clear that such $U^{(\varepsilon)}_{\alpha,n}$ exists, of course it can be infinite. Then

$$A^{(\varepsilon)}_n = \left\{\alpha \in (0,1)\ :\ n_\varepsilon^{-\frac{1}{2}}K_\varepsilon < \min\left\{\int_{z^2\in[u^2_\alpha,\infty)}|z|f_e(z)\mathrm{d}z, \alpha\right\}\right\}.$$

For $\alpha \in (0,1)\setminus A^{(\varepsilon)}_n$, i.e. for the case when

$$\min\left\{n_\varepsilon^{-\frac{1}{2}}K_\varepsilon, \int_{z^2\in[u^2_\alpha,\infty)}|z|f_e(z)\mathrm{d}z, \alpha\right\} = \min\left\{\int_{z^2\in[u^2_\alpha,\infty)}|z|f_e(z)\mathrm{d}z, \alpha\right\},$$

we can put $U^{(\varepsilon)}_{\alpha,n} = \infty$. Then of course, $u^2_\alpha < U^{(\varepsilon)}_{\alpha,n}$ and for any $\alpha \in (0,1)\setminus A^{(\varepsilon)}_n$

$$P\left(\left\{r^2_{(\ell_n)}(\beta^0) < U^{(\varepsilon)}_{\alpha,n}\right\}\right) = 1. \tag{61}$$

Moreover, (due to the fact that $U^{(\varepsilon)}_{\alpha,n} = \infty$) by the definition of u^2_α we have

$$P\left(e^2_1 \in \left[u^2_\alpha, U^{(\varepsilon)}_{\alpha,n}\right]\right) = \alpha \le n^{-\frac{1}{2}}K_\varepsilon$$

and due to (60) we have also

$$\int_{z^2\in\left[u^2_\alpha,U^{(\varepsilon)}_{\alpha,n}\right]}|z|f_e(z)\mathrm{d}z < \tfrac{1}{2}n^{-\frac{1}{2}}K^{(\varepsilon)}.$$

Let us denote by

$$C_n = \bigcap_{\alpha\in(0,1)\setminus A^{(\varepsilon)}_n}\left\{\omega\in\Omega\ :\ r^2_{(\ell_n)}(\beta^0) < U^{(\varepsilon)}_{\alpha,n}\right\}.$$

Due to (61), we have $P(C_n) = 1$. For $\alpha \in A^{(\varepsilon)}_n$, i.e. for the case when

$$\min\left\{n_\varepsilon^{-\frac{1}{2}}K_\varepsilon, \int_{z^2\in[u^2_\alpha,\infty)}|z|f_e(z)\mathrm{d}z, \alpha\right\} = n_\varepsilon^{-\frac{1}{2}}K_\varepsilon,$$

let us denote

$$v^{(\alpha)}_i = I\left\{e^2_i \le U^{(\varepsilon)}_{\alpha,n}\right\} - I\!\!E I\left\{e^2_i \le U^{(\varepsilon)}_{\alpha,n}\right\}.$$

Notice that due to the absolute continuity of the distribution function of e_i's, processes $v^{(\alpha)}_i$, $i = 1,2,...$ (where the index of process is $\alpha\in[0,1]$) are (again) separable (we shall need it later). Then, for all i (due to (60)),

$$a_n = I\!\!E I\left\{e^2_i \le U^{(\varepsilon)}_{\alpha,n}\right\} = P(e^2_i \le U^{(\varepsilon)}_{\alpha,n})$$

$$= (1-\alpha) + \min\left\{ n^{-\frac{1}{2}}K_\varepsilon, \int_{z^2\in\left[u_\alpha^2, U_{\alpha,n}^{(\varepsilon)}\right]} |z| f_e(z)\mathrm{d}z, \alpha \right\} = 1-\alpha+n^{-\frac{1}{2}}K_\varepsilon < 1, \tag{62}$$

i. e. in the case when $e_i^2 \le U_{\alpha,n}^{(\varepsilon)}$ we have $v_i^{(\alpha)} = 1 - a_n > 0$, otherwise $v_i^{(\alpha)} = -a_n < 0$. Now, let

$$\tau_{in}^{(\alpha)} = \text{time for } W(s) \text{ to exit the interval } (-a_n, 1-a_n).$$

Then applying Lemma A.1 we obtain

$$v_i^{(\alpha)} =_{\mathcal{D}} W(\tau_{in}^{(\alpha)})$$

and

$$n^{-\frac{1}{2}} \sum_{i=1}^n v_i^{(\alpha)} =_{\mathcal{D}} n^{-\frac{1}{2}} \sum_{i=1}^n W(\tau_{in}^{(\alpha)}) =_{\mathcal{D}} W(n^{-1}\sum_{i=1}^n \tau_{in}^{(\alpha)}). \tag{63}$$

Moreover, due to the fact that $(-a_n, 1-a_n) \subset (-1,1)$

$$n^{-1}\sum_{i=1}^n \tau_{in}^{(\alpha)} \le n^{-1}\sum_{i=1}^n V_i \tag{64}$$

for all $\alpha \in A_{\alpha,n}^{(\varepsilon)}$ and for $n > n_\varepsilon$. Taking into account (58), (59), (63), (64) and Lemma A.6, we have for any $n > n_\varepsilon$

$$P\left(\left\{ n^{-\frac{1}{2}} \sup_{\alpha\in A_{\alpha,n}^{(\varepsilon)}} \left|\sum_{i=1}^n v_i^{(\alpha)}\right| > \tfrac{1}{2}K_\varepsilon \right\}\right)$$

$$\le P\left(\left\{ n^{-\frac{1}{2}} \sup_{\alpha\in A_{\alpha,n}^{(\varepsilon)}} \left|\sum_{i=1}^n v_i^{(\alpha)}\right| > \tfrac{1}{2}K_\varepsilon \right\} \cap B_n^c\right) + P(B_n)$$

$$= P\left(\left\{ n^{-\frac{1}{2}} \sup_{\alpha\in A_{\alpha,n}^{(\varepsilon)}} \left|\sum_{i=1}^n W(\tau_i^{(\alpha)})\right| > \tfrac{1}{2}K_\varepsilon \right\} \cap B_n^c\right) + \tfrac{1}{4}\varepsilon$$

$$= P\left(\left\{ \sup_{\alpha\in A_{\alpha,n}^{(\varepsilon)}} \left|W(n^{-1}\sum_{i=1}^n \tau_i^{(\alpha)})\right| > \tfrac{1}{2}K_\varepsilon \right\} \cap B_n^c\right) + \tfrac{1}{4}\varepsilon$$

$$\le P\left(\left\{ \sup_{0\le s\le 2} |W(s)| > \tfrac{1}{2}K_\varepsilon \right\} \cap B_n^c\right) + \tfrac{1}{4}\varepsilon < \tfrac{1}{2}\varepsilon.$$

So we have found that for $n > n_\varepsilon$ on the set $D_n = \left\{ \omega \in \Omega \,:\, n^{-\frac{1}{2}} \sup_{\alpha\in A_{\alpha,n}^{(\varepsilon)}} \left|\sum_{i=1}^n v_i^{(\alpha)}\right| < \tfrac{1}{2}K_\varepsilon \right\}$ which has probability at least $1 - \frac{1}{2}\varepsilon$, we have for all $\alpha \in A_{\alpha,n}^{(\varepsilon)}$

$$\sum_{i=1}^n I\left\{e_i^2 \le U_{\alpha,n}^{(\varepsilon)}\right\} > \sum_{i=1}^n \mathbb{E} I\left\{e_i^2 \le U_{\alpha,n}^{(\varepsilon)}\right\} - \tfrac{1}{2} n^{\frac{1}{2}} K_\varepsilon$$

and taking into account that $\mathbb{E} I\{e_i \le U_{\alpha,n}^{(\varepsilon)}\} = 1 - \alpha + n^{-\frac{1}{2}}K_\varepsilon$ (see (62)) and recalling that $\ell_n = [(1-\alpha)n]_{int}$, we arrive at (for all $\omega \in D_n$)

$$\sum_{i=1}^n I\left\{e_i^2 \le U_{\alpha,n}^{(\varepsilon)}\right\} > n(1-\alpha) + \tfrac{1}{2}\sqrt{n}K_\varepsilon > \ell_n.$$

In other words, it means that for all $n > n_\varepsilon$ on the set D_n (which has probability at least $1 - \frac{1}{2}\varepsilon$), we have uniformly for all $\alpha \in A_{\alpha,n}^{(\varepsilon)}$ more than $\ell_n = [(1-\alpha)n]_{int}$ of the squared disturbances $e_1^2, e_2^2, ..., e_n^2$ smaller than $U_{\alpha,n}^{(\varepsilon)}$. But it implies that the ℓ_n-th order statistic among $e_1^2, e_2^2, ..., e_n^2$ is smaller than $U_{\alpha,n}^{(\varepsilon)}$.

So, we conclude that for all $n > n_\varepsilon$ ℓ_n-th order statistic among e_i^2's is smaller than $U_{\alpha,n}^{(\varepsilon)}$ simultaneously for all $\alpha \in (0,1)$ on the (same) set D_n (which has probability at least $1 - \frac{1}{2}\varepsilon$). It means that we have proved (55). (54) and (56) are fulfilled by the definition of $U_{\alpha,n}^{(\varepsilon)}$.

Similarly, we can show that there are $L_{\alpha,n}^{(\varepsilon)}$ and $\tilde{K}_\varepsilon < \infty$ such that for all $n > n_\varepsilon$

$$u_\alpha^2 \geq L_{\alpha,n}^{(\varepsilon)} \quad \text{for all} \quad \alpha \in (0,1),$$

$$P\left(\bigcap_{\alpha \in (0,1)} \left\{ \omega \in \Omega \ : \ r_{(\ell_n)}^2(\beta^0) \geq L_{\alpha,n}^{(\varepsilon)} \right\} \right) > 1 - \tfrac{1}{2}\varepsilon,$$

$$\sup_{\alpha \in (0,1)} P\left(e_i^2 \in \left[L_{\alpha,n}^{(\varepsilon)}, u_\alpha^2 \right] \right) < \tfrac{1}{2} n^{-\frac{1}{2}} \tilde{K}_\varepsilon$$

and

$$\sup_{\alpha \in (0,1)} \int_{z^2 \in \left[L_{\alpha,n}^{(\varepsilon)}, u_\alpha^2 \right]} |z| f_e(z) \mathrm{d}z < \tfrac{1}{2} n^{-\frac{1}{2}} K^{(\varepsilon)} \tag{65}$$

and the proof follows. □

In what follows for any $r, s \in R$ let us put $[r,s]_{ord} = [\min\{r,s\}, \max\{r,s\}]$.

Corollary 3 *Under assumptions of previous lemma for any $\varepsilon > 0$ there is a finite constant K_ε and $n_\varepsilon \in N$ such that for all $n > n_\varepsilon$ there is a set B_n such that $P(B_n) > 1 - \varepsilon$ and*

$$\left[e_{(\ell_n)}^2, u_\alpha^2 \right]_{ord} \cap I\{B_n\} \subset I_{\alpha,n}^{(\varepsilon)} \cap I\{B_n\} \tag{66}$$

and

$$\sup_{\alpha \in (0,1)} P\left(e_i^2 \in \left[e_{(\ell_n(\alpha))}^2, u_\alpha^2 \right]_{ord} \cap I\{B_n\} \right) \leq n^{-\frac{1}{2}} K_\varepsilon \tag{67}$$

(where $\ell_n(\alpha) = [(1-\alpha)n]_{int}$).

Proof: During the proof let us write again ℓ_n instead of $\ell_n(\alpha)$. Fix $\varepsilon \in (0,1)$. According to Lemma 6 let us find $n_\varepsilon \in N$ and $K < \infty$ so that for all $n > n_\varepsilon$ $u_\alpha^2 \in I_{\alpha,n}^{(\varepsilon)}$ and the set

$$B_n = \bigcap_{\alpha \in (0,1)} \left\{ \omega \in \Omega : r_{(\ell_n)}^2(\beta^0) \in I_{\alpha,n}^{(\varepsilon)} \right\}$$

has probability at least $1 - \frac{\varepsilon}{2}$ and

$$\sup_{\alpha \in (0,1)} P\left(e_i^2 \in I_{\alpha,n}^{(\varepsilon)} \cap I\{B_n\} \right) \leq n^{-\frac{1}{2}} K_\varepsilon.$$

Recalling that $r^2_{(\ell_n)}(\beta^0) = e^2_{(\ell_n)}$, we have $\left[e^2_{(\ell_n)}, u^2_\alpha\right]_{ord} \cap I\{B_n\} \subset I^{(\varepsilon)}_{\alpha,n} \cap I\{B_n\}$ and hence

$$\sup_{\alpha\in(0,1)} P\left(e_i^2 \in \left[e^2_{(\ell_n)}, u^2_\alpha\right]_{ord} \cap I\{B_n\}\right) \le \sup_{\alpha\in(0,1)} P\left(e_i^2 \in I^{(\varepsilon)}_{\alpha,n} \cap I\{B_n\}\right) \le n^{-\frac{1}{2}} K_\varepsilon$$

and the proof follows. □

Let us recall that we have denoted by $g(z)$ the density of d.f. $G(z) = P(e_1^2 < z)$, by u^2_α its upper α-quantile and $\ell_n(\alpha) = [(1-\alpha)n]_{int}$.

Lemma 7 *Let $\{e_i\}_{i=1}^\infty$ $(e_i \in R)$ be a sequence of independent and identically distributed random variables with absolutely continuous distribution function $F(z)$. Further, fix $\delta \in (0,1)$. Finally, let for some $\Delta = \Delta(u^2_\delta) \in (0,\infty)$*

$$\inf_{z\in(0,u^2_\delta+\Delta)} g(z) > L_g > 0. \tag{68}$$

Then for any $\varepsilon \in (0,1)$ there is a finite constant $K^{(\varepsilon)}$ and $n_\varepsilon \in N$ such that for all $n > n_\varepsilon$

$$P\left(\sup_{\alpha\in(\delta,1)} \left|r^2_{(\ell_n(\alpha))}(\beta^0) - u^2_\alpha\right| < n^{-\frac{1}{2}} \cdot K^{(\varepsilon)}\right) > 1-\varepsilon.$$

Remark 4 *Since $g(z) = \{f(\sqrt{z}) + f(-\sqrt{z})\}\, z^{-\frac{1}{2}}$, the assumption (68) is fulfilled if $f(z)$ is in a neighbourhood of zero positive. It seems to be acceptable in regression framework.*

Proof of Lemma 7. At first, we shall show that for any $\varepsilon > 0$ there is $K_\varepsilon < \infty$ so that

$$P\left(\sup_{\alpha\in(\delta,1)} \left[r^2_{(\ell_n)}(\beta^0) - u^2_\alpha\right] < n^{-\frac{1}{2}} \cdot K_\varepsilon\right) > 1-\varepsilon. \tag{69}$$

Let us fix an $\varepsilon > 0$ and denote $W(s)$ the Wiener process. Further, let us find $K^{(1)} < \infty$ such that

$$P\left(\sup_{0\le s\le 2} |W(s)| > \tfrac{1}{2}K^{(1)}\right) < \tfrac{1}{2}\varepsilon. \tag{70}$$

Moreover, let $K^{(\varepsilon)} = K^{(1)} L_g^{-1}$ and $n_\Delta = 2 \cdot [K^{(\varepsilon)}\Delta^{-1}]_{int} + 1$. In what follows, let us restrict ourselves on $n > n_\delta$. Finally, let us denote

$$v_i^{(\alpha)} = I\left\{e_i^2 \le u^2_\alpha + n^{-\frac{1}{2}}K^{(\varepsilon)}\right\} - I\!E I\left\{e_i^2 \le u^2_\alpha + n^{-\frac{1}{2}}K^{(\varepsilon)}\right\}. \tag{71}$$

Then, for all i,

$$I\!E I\left\{e_i^2 \le u^2_\alpha + n^{-\frac{1}{2}}K^{(\varepsilon)}\right\} = P(e_i^2 \le u^2_\alpha + n^{-\frac{1}{2}}K^{(\varepsilon)})$$
$$= a_n \ge (1-\alpha) + n^{-\frac{1}{2}}K^{(\varepsilon)}L_g > (1-\alpha) + n^{-\frac{1}{2}}K^{(1)} > 1-\alpha, \tag{72}$$

i. e. in the case when $e_i^2 \le u^2_\alpha + n^{-\frac{1}{2}}K^{(\varepsilon)}$ we have $v_i^{(\alpha)} = 1 - a_n > 0$ otherwise $v_i^{(\alpha)} = -a_n < 0$. Now, let

$$\tau^{(\alpha)}_{in} = \text{time for } W(s) \text{ to exit the interval } (-a_n, 1-a_n).$$

Then applying Lemma A.1 we obtain

$$v_i^{(\alpha)} =_{\mathcal{D}} W(\tau_{in}^{(\alpha)})$$

and

$$n^{-\frac{1}{2}} \sum_{i=1}^{n} v_i^{(\alpha)} =_{\mathcal{D}} n^{-\frac{1}{2}} \sum_{i=1}^{n} W(\tau_{in}^{(\alpha)}) =_{\mathcal{D}} W(n^{-1} \sum_{i=1}^{n} \tau_{in}^{(\alpha)}). \tag{73}$$

Further, let us define a sequence of i.i.d. r.v. $\{V_i\}_{i=1}^{\infty}$

$$V_i = \text{time for } W(s) \text{ to exit the interval } (-1, 1)$$

and applying Lemma A.1 once again we have $I\!EV_i = 1$ for all i. Moreover,

$$n^{-1} \sum_{i=1}^{n} \tau_{in}^{(\alpha)} \leq n^{-1} \sum_{i=1}^{n} V_i \tag{74}$$

for all $\alpha \in (\delta, 1)$ and for $n > n_\varepsilon$. Now, for $\varepsilon > 0$ (which was fixed at the beginning of the proof) let us find $n_\varepsilon > n_\delta$ such that for any $n > n_\varepsilon$, putting

$$B_n = \left\{ \omega \in \Omega \ : \ n^{-1} \sum_{i=1}^{n} V_i > 2 \right\},$$

we have

$$P(B_n) < \tfrac{1}{2}\varepsilon. \tag{75}$$

Taking into account (70), (73), (74), (75) and again Lemma A.6, we have for any $n > n_\varepsilon$

$$P\left(\left\{ n^{-\frac{1}{2}} \sup_{\alpha \in (\delta,1)} \left| \sum_{i=1}^{n} v_i^{(\alpha)} \right| > \tfrac{1}{2} K^{(1)} \right\} \right)$$

$$\leq P\left(\left\{ n^{-\frac{1}{2}} \sup_{\alpha \in (\delta,1)} \left| \sum_{i=1}^{n} v_i^{(\alpha)} \right| > \tfrac{1}{2} K^{(1)} \right\} \cap B_n^c \right) + P\left(B_n\right)$$

$$= P\left(\left\{ n^{-\frac{1}{2}} \sup_{\alpha \in (\delta,1)} \left| \sum_{i=1}^{n} W(\tau_i^{(\alpha)}) \right| > \tfrac{1}{2} K^{(1)} \right\} \cap B_n^c \right) + \tfrac{1}{2}\varepsilon$$

$$= P\left(\left\{ \sup_{\alpha \in (\delta,1)} \left| W(n^{-1} \sum_{i=1}^{n} \tau_i^{(\alpha)}) \right| > \tfrac{1}{2} K^{(1)} \right\} \cap B_n^c \right) + \tfrac{1}{2}\varepsilon$$

$$\leq P\left(\left\{ \sup_{s \leq 2} \ |W(s)| > \tfrac{1}{2} K^{(1)} \right\} \cap B_n^c \right) + \tfrac{1}{2}\varepsilon < \varepsilon.$$

So we have found that for $n > n_\varepsilon$ the set $\left\{ n^{-\frac{1}{2}} \sup_{\alpha \in (\delta,1)} \left| \sum_{i=1}^{n} v_i^{(\alpha)} \right| \leq \frac{1}{2} K^{(1)} \right\}$ has probability at least $1 - \varepsilon$. In other words, we have for all $\alpha \in (\delta, 1)$ with probability at least $1 - \varepsilon$

$$\sum_{i=1}^{n} I\left\{ e_i^2 \leq u_\alpha^2 + n^{-\frac{1}{2}} K^{(\varepsilon)} \right\} > \sum_{i=1}^{n} I\!EI\left\{ e_i^2 \leq u_\alpha^2 + n^{-\frac{1}{2}} K^{(\varepsilon)} \right\} - \tfrac{1}{2} n^{\frac{1}{2}} K^{(1)}$$

and taking into account (72) and recalling that $\ell_n = [(1-\alpha)n]_{int}$, we arrive at

$$\sum_{i=1}^{n} I\left\{e_i^2 \leq u_\alpha^2 + n^{-\frac{1}{2}}K^{(\varepsilon)}\right\} > n(1-\alpha) + \tfrac{1}{2}\sqrt{n}K^{(1)} > \ell_n.$$

In other words, it means that for all $n > n_\varepsilon$ on the set of probability at least $1-\varepsilon$, we have uniformly for all $\alpha \in (\delta, 1)$ more than $\ell_n = [(1-\alpha)n]_{int}$ of the squared disturbances e_i^2's smaller than $u_\alpha^2 + n^{-\frac{1}{2}}K^{(\varepsilon)}$. But it implies that the ℓ_n-th order statistic among e_i^2's is smaller than $u_\alpha^2 + n^{-\frac{1}{2}}K^{(\varepsilon)}$. Since it holds for all $n > n_\varepsilon$ and simultaneously for all $\alpha \in (\delta, 1)$ on the (same) set of probability at least $1-\varepsilon$, we have proved (69). The probability of the lower inequality may be found considering

$$\tilde{v}_i^{(\alpha)} = I\left\{e_i^2 \geq u_\alpha^2 - n^{-\frac{1}{2}}K^{(\varepsilon)}\right\} - I\!\!E I\left\{e_i^2 \geq u_\alpha^2 - n^{-\frac{1}{2}}K^{(\varepsilon)}\right\}. \qquad \square$$

Remark 5 *Let us stress that we were able to prove that the α-quantile and the corresponding order statistic are close each to other "uniformly" in $\alpha \in (\delta, 1)$. At the first glance it may seem to be rather strong result but it is probably very "natural". We may see the situation as follows. We have assumed a sequence $\{e_i\}_{i=1}^{\infty}$ $(e_i \in R)$ of independent and identically distributed random variables. We may assume that each of these variables is defined on a probability space $\Omega_i, \mathcal{A}_i, P_i$ (which are copies of one fix space) and the basic probability space (we have mentioned at the beginning of the paper) is then the infinite Carthesian product of these spaces with probability defined in the way which is used when proving Kolmogorov extension theorem, see e. g. Breiman (1968). We may also assume that on each of these spaces we have defined Wiener process. Then the system of sequences of the random variables given as the stopping times $\left\{\tau_{in}^{(\alpha)}\right\}_{i=1}^{n}$, $n = 1, 2, ..., \alpha \in (\delta, 1)$ is such that for any couple $\alpha_1 \leq \alpha_2$, any $i = 1, 2, .., n$, any $n \in N$ and any $\omega \in \Omega$ we have $\tau_{in}^{(\alpha_1)} \geq \tau_{in}^{(\alpha_2)}$ since*

$$I\left\{e_i^2 \leq u_{\alpha_1}^2 + n^{-\frac{1}{2}}K^{(\varepsilon)}\right\} \geq I\left\{e_i^2 \leq u_{\alpha_2}^2 + n^{-\frac{1}{2}}K^{(\varepsilon)}\right\}$$

and hence all intervals, for α starting from δ to 1, are nested. So we model the sequence $\left\{v_i^{(\alpha)}\right\}_{i=1}^{n}$, $n = 1, 2, ..., \alpha \in (\delta, 1)$ (see (71)) by the same Wiener processes but the lengths of the intervals when we stop them decrease with increasing α. So in fact the resulting set (for each n) for all $\alpha \in (\delta, 1)$ is given by the set which corresponds to the sequence of $\left\{v_i^{(\delta)}\right\}_{i=1}^{\infty}$, see again (71).

Lemma 8 *Let $\{e_i\}_{i=1}^{\infty}$ $(e_i \in R)$ be a sequence of independent and identically distributed random variables with absolutely continuous distribution function $F(z)$ and denote the corresponding density by $f_e(x)$. Finally, fix $\delta \in (0, 1)$ and let for some $\Delta = \Delta(u_\delta^2) \in (0, \infty)$*

$$\inf_{z \in (0, u_\delta^2 + \Delta)} f(z) > L_f > 0.$$

Then for any $\varepsilon \in (0, 1)$ there is a finite constant $K^{(\varepsilon)}$ and $n_\varepsilon \in N$ such that for all $n > n_\varepsilon$

$$P\left(\sup_{\alpha \in (\delta, 1)} \left|\sqrt{r^2_{(\ell_n(\alpha))}(\beta^0)} - u_\alpha\right| < n^{-\frac{1}{2}} \cdot K^{(\varepsilon)}\right) > 1 - \varepsilon.$$

Proof runs nearly along the same line as the proof of previous lemma.

Let us recall once again that we have denoted for any $\alpha \in (0,1)$ $\ell_n(\alpha) = [(1-\alpha)n]_{int}$ and that we have denoted by $g(z)$ the density of d.f. $G(z) = P(e_1^2 < z)$.

Lemma 9 *Let $\{e_i\}_{i=1}^{\infty}$ ($e_i \in R$) be a sequence of independent and identically distributed random variables with absolutely continuous distribution function $F(z)$ and denote the corresponding density by $f_e(z)$. Moreover, let $f(z)$ be bounded (say by U_f) and uniformly locally Lipschitz of the first order in z, so that there is a finite constant K_f such that there is a positive τ so that for any pair $z_1, z_2 \in R$, $|z_1 - z_2| < \tau$ we have $|f(z_1) - f(z_2)| \leq K_f \cdot |z_1 - z_2|$. Finally, fix $\delta \in (0,1)$ and let for some $\Delta = \Delta(u_\delta^2) \in (0,\infty)$*

$$\inf_{z \in (0, u_\delta^2 + \Delta)} g(z) > L_g > 0 \quad \text{and} \quad \inf_{z \in (0, \sqrt{u_\delta^2 + \Delta})} f(z) > L_f > 0. \tag{76}$$

Then for any $\varepsilon > 0$ there is a finite constant $K^{(\varepsilon)}$ and $n_\varepsilon \in N$ such that for any $n > n_\varepsilon$ there is a set D_n such that $P(D_n) > 1 - \varepsilon$,

$$\max_{1 \leq i \leq n} \sup_{\alpha \in (\delta,1)} \left| I\!\!E \left\{ I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e_{(\ell_n(\alpha))}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell_n(\alpha)}{n}}^2\} \right] \cdot I\{D_n\} \right\} \right| < n^{-\frac{3}{2}} \cdot K^{(\varepsilon)} \tag{77}$$

and

$$\max_{1 \leq i \leq n} \sup_{\alpha \in (\delta,1)} \left| I\!\!E \left\{ I\{e_i < 0\} \cdot \left[I\{e_i^2 \leq e_{(\ell_n(\alpha))}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell_n(\alpha)}{n}}^2\} \right] \cdot I\{D_n\} \right\} \right| < n^{-\frac{3}{2}} \cdot K^{(\varepsilon)}. \tag{78}$$

Proof: We are going to consider at first (77). Let us fix $\varepsilon > 0$. Employing Lemmas 7 and 8, find $K_{\varepsilon,1} < \infty$ and $n_{\varepsilon,1} \in N$ such that for all $n > n_{\varepsilon,1}$, putting

$$B_n^{(1)} = \left\{ \sup_{\alpha \in (\delta,1)} \left| \sqrt{e_{(\ell_n(\alpha))}^2} - u_\alpha \right| < n^{-\frac{1}{2}} \cdot K_{\varepsilon,1} \right\} \tag{79}$$

and

$$B_n^{(2)} = \left\{ \sup_{\alpha \in (\delta,1)} \left| e_{(\ell_n(\alpha))}^2 - u_\alpha^2 \right| < n^{-\frac{1}{2}} \cdot K_{\varepsilon,1} \right\}, \tag{80}$$

the set $C_n = B_n^{(1)} \cap B_n^{(2)}$ has probability

$$P(C_n) > 1 - \varepsilon.$$

(In what follows we shall write mostly ℓ instead of $\ell_n(\alpha)$, if no misinterpretation can occur.) Further, let us denote $D_{i,\ell,n} = \left\{ \omega \in \Omega : e_i(\omega) = e_{(\ell)}(\omega) \right\}$. Then Assertion A.1 implies that $P(D_{i,\ell,n}) = \frac{1}{n}$. Since

$$\left| I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right] \cdot I\{C_n\} \right| \leq 1,$$

we have

$$I\!\!E \left| I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right] \cdot I\{C_n\} \cdot I\{D_{i,\ell,n}\} \right| \leq I\!\!E I\{D_{i,\ell,n}\} \leq \frac{1}{n},$$

hence

$$\left|\mathbb{E}\left\{I\{e_i>0\}\cdot\left[I\{e_i^2\le e_{(\ell)}^2\}-I\{e_i^2\le u_{1-\frac{\ell}{n}}^2\}\right]\cdot I\{C_n\}\right\}\right|$$

$$=\left|\mathbb{E}\left\{I\{e_i>0\}\cdot\left[I\{e_i^2\le e_{(\ell)}^2\}-I\{e_i^2\le u_{1-\frac{\ell}{n}}^2\}\right]\cdot I\{C_n\}\cdot I\{D_{i,\ell,n}^c\}\right\}\right.$$

$$\left.+\,\mathbb{E}\left\{I\{e_i>0\}\cdot\left[I\{e_i^2\le e_{(\ell)}^2\}-I\{e_i^2\le u_{1-\frac{\ell}{n}}^2\}\right]\cdot I\{C_n\}\cdot I\{D_{i,\ell,n}\}\right\}\right|$$

$$\le\left|\mathbb{E}\left\{I\{e_i>0\}\cdot\left[I\{e_i^2\le e_{(\ell)}^2\}-I\{e_i^2\le u_{1-\frac{\ell}{n}}^2\}\right]\cdot I\{C_n\}\cdot I\{D_{i,\ell,n}^c\}\right\}\right|+\frac{1}{n}.$$

Let us recall that

$$\left|\mathbb{E}\left\{I\{e_i>0\}\cdot\left[I\{e_i^2\le e_{(\ell)}^2\}-I\{e_i^2\le u_{1-\frac{\ell}{n}}^2\}\right]\cdot I\{C_n\}\cdot I\{D_{i,\ell,n}^c\}\right\}\right|$$

$$=\left|\mathbb{E}_{\sqrt{e_{(\ell)}^2}}\left\{\mathbb{E}\left(I\{e_i>0\}\cdot\left[I\{e_i^2\le e_{(\ell)}^2\}-I\{e_i^2\le u_{1-\frac{\ell}{n}}^2\}\right]\cdot I\{C_n\}\cdot I\{D_{i,\ell,n}^c\}\,\middle|\,\sqrt{e_{(\ell)}^2}=z_1^2\right)\right\}\right|$$

where the subindex of $\mathbb{E}_{\sqrt{e_{(\ell)}^2}}$ indicates that it is the mean value over $\sqrt{e_{(\ell)}^2}=z_1$. We are going to find at first bounds of

$$\mathbb{E}\left(I\{e_i>0\}\cdot\left[I\{e_i^2\le e_{(\ell)}^2\}-I\{e_i^2\le u_{1-\frac{\ell}{n}}^2\}\right]\cdot I\{C_n\}\cdot I\{D_{i,\ell,n}^c\}\,\middle|\,\sqrt{e_{(\ell)}^2}=z_1\right). \tag{81}$$

First of all, we are going to show that when evaluating the conditional mean value (81), due to presence of $I\{C_n\}$ in integrand, we need to do it only for z_1 from a subinterval of

$$\left(0,u_\delta+n^{-\frac{1}{2}}\cdot K_{\varepsilon,1}\right). \tag{82}$$

Keep in mind that for any $n>n_{\varepsilon,1}$ and $\omega\in C_n$

$$\left|\sqrt{e_{(\ell_n(\alpha))}^2}-u_{1-\frac{\ell_n(\alpha)}{n}}\right|<n^{-\frac{1}{2}}\cdot K_{\varepsilon,1}$$

and hence for any $n>n_{\varepsilon,1}$ and $\omega\in C_n$ we have

$$\left(\sqrt{e_{(\ell_n(\alpha))}^2},u_{1-\frac{\ell_n(\alpha)}{n}}\right)_{ord}\subset\left(0,u_{1-\frac{\ell_n(\alpha)}{n}}+n^{-\frac{1}{2}}\cdot K_{\varepsilon,1}\right). \tag{83}$$

As for any $\alpha\in(\delta,1)$ we have $u_\alpha<u_\delta$, for $b=u_\delta+\Delta(\delta)$ (for $\Delta(\delta)$ see Lemmas 7 and 8) let us find an $n_{\varepsilon,2}$ so that for any $n>n_{\varepsilon,2}$ and $\alpha\in(\delta,1)$

$$\left(\sqrt{e_{(\ell_n(\alpha))}^2},u_{1-\frac{\ell_n(\alpha)}{n}}\right)_{ord}\subset(0,b). \tag{84}$$

Secondly,

$$I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e_{(\ell)}^2\} - I\{e_i^2 \le u_{1-\frac{\ell}{n}}^2\} \right] \cdot I\{C_n\} \cdot I\{D_{i,\ell,n}^c\} = 1 \tag{85}$$

iff $u_{1-\frac{\ell}{n}} \le e_i < \sqrt{e_{(\ell)}^2}$ and

$$I\{e_i > 0\} \cdot \left[I\{e_i^2 < e_{(\ell)}^2\} - I\{e_i^2 \le u_{1-\frac{\ell}{n}}^2\} \right] \cdot I\{C_n\} \cdot I\{D_{i,\ell,n}^c\} = -1 \tag{86}$$

iff $\sqrt{e_{(\ell)}^2} \le e_i < u_{1-\frac{\ell}{n}}$. Now let us find the conditional density of e_i given $\sqrt{e_{(\ell)}^2}$ and analogously as we have denoted by $f_{e|X}(v|X_1 = x)$ the conditional density of e_1 given X_1, let us denote this conditional density by $f_{e_i \left| \sqrt{e_{(\ell)}^2} \right.}(v \mid \sqrt{e_{(\ell)}^2} = z_1)$. Of course, the conditional density is ratio of the joint density of e_i and $\sqrt{e_{(\ell)}^2}$ divided by the density of $\sqrt{e_{(\ell)}^2}$. Let us start with the latter and denote $H(z)$ the d. f. and $h(z)$ the density of r. v. $|e_1|$ (please, see at this moment the proof of Lemma A.4, the relation(A.3)). First of all, let us find the probability that at least ℓ r.v.'s are smaller than z_1. We select a group of r.v.'s containing k of them ($k \ge \ell$) from n r.v.'s. Then the searched probability is equal to

$$\sum_{k=\ell}^{n} \frac{n!}{k!(n-k)!} H^k(z_1)) \left[1 - H(z_1)\right]^{n-k}.$$

Hence the density of $\sqrt{e_{(\ell)}^2}$ is (see (A.4))

$$\frac{n!}{(\ell-1)!(n-\ell)!} H^{\ell-1}(z_1) \left[1 - H(z_1)\right]^{n-\ell} h(z_1). \tag{87}$$

Now, let us find the joint density of e_i and $\sqrt{e_{(\ell)}^2}$. Remember that the integrand in (81) contains $I\{e_i > 0\}$, i. e. we assume that $e_i > 0$. First of all, we shall find the probability that $\sqrt{e_{(\ell)}^2} < z_1$ and $0 < e_i < z_2$. Firstly, let us assume that $z_1 > z_2$. Then, similarly as in previous, we select from $n-1$ r.v.'s (namely, $|e_1|, |e_2|, ..., |e_{i-1}|, |e_{i+1}|, ..., |e_n|$) a group containing at least $k \ge \ell - 1$ r.v.'s. The corresponding probability is then

$$\sum_{k=\ell-1}^{n} \frac{(n-1)!}{k!(n-1-k)!} H^k(z_1)) \cdot \left[1 - H(z_1)\right]^{n-1-k} \cdot \left[F(z_2) - F(0)\right]$$

and hence the joint density is given as

$$\frac{(n-1)!}{(\ell-2)!(n-\ell)!} H^{\ell-2}(z_1) \cdot \left[1 - H(z_1)\right]^{n-\ell} \cdot h(z_1) \cdot f(z_2).$$

Finally, the conditional density (for the case when $z_1 > z_2$)

$$f_{e_i \left| \sqrt{e_{(\ell)}^2} \right.}(z_2 \mid \sqrt{e_{(\ell)}^2} = z_1) = \frac{\ell - 2}{n} H^{-1}(z_1) \cdot f(z_2). \tag{88}$$

Secondly, let $z_1 < z_2$. Along the same lines as in previous we arrive at

$$\sum_{k=\ell}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} H^k(z_1)) \cdot \left[1 - H(z_1)\right]^{n-1-k} \cdot \left[F(z_2) - F(z_1)\right]$$

and hence the joint density is given now as

$$\frac{(n-1)!}{(\ell-1)!(n-1-\ell)!} H^{\ell-1}(z_1) \cdot [1-H(z_1)]^{n-1-\ell} \cdot h(z_1) \cdot f(z_2).$$

So, the conditional density is (for the case when $z_1 < z_2$)

$$f_{e_i \big| \sqrt{e^2_{(\ell)}}}(z_2 \mid \sqrt{e^2_{(\ell)}} = z_1) = \frac{n-\ell}{n} H^{-1}(z_1) \cdot f(z_2). \tag{89}$$

Finally, due to the presence of $I\{D^c_{i,\ell,n}\}$ in (81), we restrict ourselves on the case when

$$\left\{\omega \in \Omega : e_i(\omega) \neq e_{(\ell)}(\omega)\right\}$$

we can neglect finding the conditional density for $e_i(\omega) = e_{(\ell)}$. Now, it is clear that due to (84), i. e. due to the fact that we need to evaluate conditional mean value (77) only for $z_1 \in (0, b)$, and the assumptions (76), both

$$\frac{\ell-2}{n} H^{-1}(z_1) \qquad \text{as well as} \qquad \frac{n-\ell}{n} H^{-1}(z_1)$$

are uniformly in n, ℓ and z_1 bounded by a finite constant. It implies, again due to the assumptions of lemma (that $f(z)$ is uniformly locally Lipschitz in z), the conditional density $f_{e_i \big| \sqrt{e^2_{(\ell)}}}(z \mid \sqrt{e^2_{(\ell)}} = z_1)$ is uniformly locally Lipschitz in z. It means that there is a finite constant $\tilde{K}_f$ and $\tau > 0$ so that for any pair $z^*, z^{**} \in R$ such that $|z^* - z^{**}| < \tau$ and any $\ell = 1, 2, ..., \ell_n(\delta)$ we have $\left| f_{e_i \big| \sqrt{e^2_{(\ell)}}}(z^* \mid \sqrt{e^2_{(\ell)}} = z_1) - f_{e_i \big| \sqrt{e^2_{(\ell)}}}(z^{**} \mid \sqrt{e^2_{(\ell)}} = z_1) \right| < \tilde{K}_f \cdot |z^* - z^{**}|$. It implies that there is $n_{\varepsilon,3} > n_{\varepsilon,2}$ such that for any $n > n_{\varepsilon,3}$, any $\ell = 1, 2, ..., \ell_n(\delta)$ (or equivalently for any $\alpha \in (\delta, 1)$) and any $z, |z - u_\alpha| < n^{-\frac{1}{2}} K_{\varepsilon,1}$, we have $\left| f_{e_i \big| \sqrt{e^2_{(\ell)}}}(z \mid \sqrt{e^2_{(\ell)}} = z_1) - f_{e_i \big| \sqrt{e^2_{(\ell)}}}(u_\alpha \mid \sqrt{e^2_{(\ell)}} = z_1) \right| < \tilde{K}_f \cdot |z - u_\alpha|$.

Then if $\sqrt{e^2_{(\ell)}} = z_1 < u_{1-\frac{\ell}{n}}$, we have (see (86))

$$\mathbb{E}\left(I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e^2_{(\ell)}\} - I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} \right] \cdot I\{C_n\} \cdot I\{D^c_{i,\ell,n}\} \middle| \sqrt{e^2_{(\ell)}} = z_1 \right).$$

$$\begin{aligned}
&= -\int_{z_1}^{u_{1-\frac{\ell}{n}}} f_{e_i \big| \sqrt{e^2_{(\ell)}}}(z_2 \mid \sqrt{e^2_{(\ell)}} = z_1) \mathrm{d}z_2 = -\int_{z_1}^{u_{1-\frac{\ell}{n}}} f_{e_i \big| \sqrt{e^2_{(\ell)}}}(u_{1-\frac{\ell}{n}} \mid \sqrt{e^2_{(\ell)}} = z_1) \mathrm{d}z_2 \\
&\quad - \int_{z_1}^{u_{1-\frac{\ell}{n}}} \left\{ f_{e_i \big| \sqrt{e^2_{(\ell)}}}(z_2 \mid \sqrt{e^2_{(\ell)}} = z_1) - f_{e_i \big| \sqrt{e^2_{(\ell)}}}(u_{1-\frac{\ell}{n}} \mid \sqrt{e^2_{(\ell)}} = z_1) \right\} \mathrm{d}z_2 \\
&= -f_{e_i \big| \sqrt{e^2_{(\ell)}}}(u_{1-\frac{\ell}{n}} \mid \sqrt{e^2_{(\ell)}} = z_1) \left(u_{1-\frac{\ell}{n}} - z_1 \right) + R_{n1}(z_1)
\end{aligned} \tag{90}$$

where, due to the fact that

$$\left| f_{e_i \big| \sqrt{e^2_{(\ell)}}}(z_2 \mid \sqrt{e^2_{(\ell)}} = z_1) - f_{e_i \big| \sqrt{e^2_{(\ell)}}}(u_{1-\frac{\ell}{n}} \mid \sqrt{e^2_{(\ell)}} = z_1) \right| \leq \widetilde{K}_f \cdot \left| z_2 - u_{1-\frac{\ell}{n}} \right| \leq \widetilde{K}_f \cdot \left| z_1 - u_{1-\frac{\ell}{n}} \right|,$$

we have

$$|R_{n1}(z_1)| \le \widetilde{K}_f \cdot \left| z_1 - u_{1-\frac{\ell}{n}} \right| \int_{z_1}^{u_{1-\frac{\ell}{n}}} \le \tilde{K}_f \cdot \left[u_{1-\frac{\ell}{n}} - z_1 \right]^2. \tag{91}$$

Hence for any $n > n_{\varepsilon,1}$ and any $\omega \in C_n$ (see (79))

$$|R_{n1}(z_1)| \le n^{-1} \cdot \widetilde{K}_f \cdot K_{\varepsilon,1}^2. \tag{92}$$

For $u_{1-\frac{\ell}{n}} < z_1 = \sqrt{e_{(\ell)}^2}$, we have (see (85))

$$I\!E \left(I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e_{(\ell)}^2\} - I\{e_i^2 \le u_{1-\frac{\ell}{n}}^2\} \right] \cdot I\{C_n\} \cdot I\{D_{i,\ell,n}^c\} \middle| \sqrt{e_{(\ell)}^2} = z_1^2 \right).$$

$$= f_{e_i \left| \sqrt{e_{(\ell)}^2} \right.}(u_{1-\frac{\ell}{n}} \mid \sqrt{e_{(\ell)}^2} = z_1) + R_{n2}(z_1) \tag{93}$$

with of course

$$|R_{n2}(z_1)| < \tilde{K}_f \cdot \left[u_{1-\frac{\ell}{n}} - z_1 \right]^2 \tag{94}$$

and hence again for any $n > n_{\varepsilon,1}$ and any $\omega \in C_n$

$$|R_{n2}(z_1)| \le n^{-1} \cdot \widetilde{K}_f \cdot K_{\varepsilon,1}^2. \tag{95}$$

Now employing Corollary A.1, we can find $K_{\varepsilon,2} < \infty$ and $n_{\varepsilon,4} > n_{\varepsilon,3}$ so that for any $n > n_{\varepsilon,4}$ we have for the density of $\hat{u}_{1-\frac{\ell}{n}} = \sqrt{e_{(\ell_n(\alpha))}^2}$(see (A.9))

$$h_{n,\alpha}(u) = h_{n,\alpha}^*(u) + \varrho_{n,\alpha}(u)$$

with

$$\sup_{\alpha \in (\delta,1)} \sup_{|u| \le K_{\varepsilon,1}} |\varrho_{n,\alpha}(u)| \le n^{-\frac{1}{2}} \cdot K_{\varepsilon,2}$$

(for $\varrho_{n,\alpha}(u)$ see (A.9) and (A.10) and notice that we have put $K = K_{\varepsilon,1}$ in (A.10)) and $h_{n,\alpha}^*(u)$ is uniformly in $n \in N$, $\alpha \in (0,1)$ and $u \in R$ bounded by a finite constant say U_h. Now we have

$$I\!E_{\sqrt{e_{(\ell)}^2}} \left\{ I\!E \left(I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e_{(\ell)}^2\} - I\{e_i^2 \le u_{1-\frac{\ell}{n}}^2\} \right] \cdot I\{C_n\} \cdot I\{D_{i,\ell,n}^c\} \middle| \sqrt{e_{(\ell)}^2} = z_1 \right) \right\}$$

$$= - \int_{u_{1-\frac{\ell}{n}} - n^{-\frac{1}{2}} \cdot K_{\varepsilon,2}}^{u_{1-\frac{\ell}{n}}} \left[f_{e_i \left| \sqrt{e_{(\ell)}^2} \right.}(u_{1-\frac{\ell}{n}} \mid \sqrt{e_{(\ell)}^2} = z_1) \left(u_{1-\frac{\ell}{n}} - z_1 \right) + R_{n1}(z_1) \right] \cdot \left[h_{n,\alpha}^*(z_1) + \varrho_{n,\alpha}(z_1) \right] \mathrm{d}z_1$$

$$+ \int_{u_{1-\frac{\ell}{n}}}^{u_{1-\frac{\ell}{n}} + n^{-\frac{1}{2}} \cdot K_{\varepsilon,2}} \left[f_{e_i \left| \sqrt{e_{(\ell)}^2} \right.}(u_{1-\frac{\ell}{n}} \mid \sqrt{e_{(\ell)}^2} = z_1) \left(z_1 - u_{1-\frac{\ell}{n}} \right) + R_{n2}(z_1) \right] \cdot \left[h_{n,\alpha}^*(z_1) + \varrho_{n,\alpha}(z_1) \right] \mathrm{d}z_1$$

Now, for any pair $z_1^* \in \left[u_{1-\frac{\ell}{n}} - n^{-\frac{1}{2}} \cdot K_{\varepsilon,2}, u_{1-\frac{\ell}{n}} \right]$ and $z_1^{**} \in \left[u_{1-\frac{\ell}{n}}, u_{1-\frac{\ell}{n}} + n^{-\frac{1}{2}} \cdot K_{\varepsilon,2} \right]$ such that $u_{1-\frac{\ell}{n}} - z_1^* = z_1^{**} - u_{1-\frac{\ell}{n}}$ we have

$$f_{e_i \left| \sqrt{e_{(\ell)}^2} \right.}(u_{1-\frac{\ell}{n}} \mid \sqrt{e_{(\ell)}^2} = z_1) \left(u_{1-\frac{\ell}{n}} - z_1^* \right) \cdot h_{n,\alpha}^*(z_1^*)$$

$$= f_{e_i \big| \sqrt{e^2_{(\ell)}}}(u_{1-\frac{\ell}{n}} \mid \sqrt{e^2_{(\ell)}} = z_1) \left(z_1^{**} - u_{1-\frac{\ell}{n}}\right) \cdot h^*_{n,\alpha}(z_1^{**}).$$

Taking into account (90), (92), (93) and (95), we arrive at

$$\left| I\!E_{\sqrt{e^2_{(\ell)}}} \left\{ I\!E \left(I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right] \cdot I\{C_n\} \cdot I\{D^c_{i,\ell,n}\} \Big| \sqrt{e^2_{(\ell)}} = z_1 \right) \right\} \right|$$

$$\le \left| \int_{u_{1-\frac{\ell}{n}} - n^{-\frac{1}{2}} K_{\varepsilon,1}}^{u_{1-\frac{\ell}{n}}} f_{e_i \big| \sqrt{e^2_{(\ell)}}}(u_{1-\frac{\ell}{n}} \mid \sqrt{e^2_{(\ell)}} = z_1) \left(u_{1-\frac{\ell}{n}} - z_1 \right) \cdot \varrho_{n,\alpha}(z_1) \mathrm{d}z_1 \right|$$

$$+ \left| \int_{u_{1-\frac{\ell}{n}} - n^{-\frac{1}{2}} K_{\varepsilon,1}}^{u_{1-\frac{\ell}{n}}} R_{n1}(z_1) \cdot \left[U_h + \varrho_{n,\alpha}(z_1) \right] \mathrm{d}z_1 \right|$$

$$+ \left| \int_{u_{1-\frac{\ell}{n}}}^{u_{1-\frac{\ell}{n}} + n^{-\frac{1}{2}} K_{\varepsilon,1}} f_{e_i \big| \sqrt{e^2_{(\ell)}}}(u_{1-\frac{\ell}{n}} \mid \sqrt{e^2_{(\ell)}} = z_1) \left(u_{1-\frac{\ell}{n}} - z_1 \right) \cdot \varrho_{n,\alpha}(z_1) \mathrm{d}z_1 \right|$$

$$+ \left| \int_{u_{1-\frac{\ell}{n}} - n^{-\frac{1}{2}} K_{\varepsilon,1}}^{u_{1-\frac{\ell}{n}}} R_{n2}(z_1) \cdot \left[U_h + \varrho_{n,\alpha}(z_1) \right] \mathrm{d}z_1 \right|$$

$$\le \left\{ U_f \cdot n^{-\frac{1}{2}} \cdot K_{\varepsilon,1} \cdot n^{-\frac{1}{2}} \cdot K_{\varepsilon,2} + n^{-\frac{1}{2}} \cdot \tilde{K}_f \cdot K_{\varepsilon,1} \cdot \left[U_h + n^{-\frac{1}{2}} \cdot K_{\varepsilon,2} \right] \right\} \times$$

$$\times \left\{ \int_{u_{1-\frac{\ell}{n}} - n^{-\frac{1}{2}} K_{\varepsilon,1}}^{u_{1-\frac{\ell}{n}}} + \int_{u_{1-\frac{\ell}{n}}}^{u_{1-\frac{\ell}{n}} + n^{-\frac{1}{2}} K_{\varepsilon,1}} \right\} \mathrm{d}z_1.$$

Then we conclude that there is a finite constant $C^{(1)}$ such that

$$\sup_{\alpha \in (\delta,1)} \left| I\!E \left(I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right] \cdot I\{C_n\} \cdot I\{D^c_{i,\ell,n}\} \right) \right| \le n^{-\frac{3}{2}} \cdot C^{(1)}.$$

Along similar lines we prove also (78). That concludes the proof. □

We shall need some other assumptions.

AC1 For any $a \in R^+$ there is $\Delta(a) > 0$ so that

$$\inf_{z \in (0, a+\Delta(a))} g(z) > L_{g,a} > 0.$$

AC2 There is $q > 1$ so that $I\!E\,|e_1|^{2q} < \infty$.

Lemma 10 *Let the conditions* **C1**, **C2**, **C3**, **C4**, **NC1**, **NC2**, **AC1** *and* **AC2** *hold. Then*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n w\left(F_{\beta^0}(|e_i|)\right) \cdot Z_i e_i = \frac{1}{n} \sum_{i=1}^n w\left(F_{\beta^0}(|e_i|)\right) \cdot Z_i X_i' \cdot \left\{ \sqrt{n} \left(\hat{\beta}^{(IWV,n,w)} - \beta^0 \right) \right\} + o_p(1). \quad (96)$$

Remark 6 *Although we have assumed $\beta^0 = 0$, we have included β^0 in (96) because the usual form of asymptotic representation is (probably always) given with β^0.*

Proof of Lemma 10: Returning to (49), let us write

$$\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_i \left(Y_i - X_i'\beta^0\right) I\{e_i^2 \leq e_{(\ell)}^2\} \tag{97}$$

$$= \frac{1}{\sqrt{n}}\sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_i X_i' \left(\hat{\beta}^{(IWV,n,w)} - \beta^0\right) \cdot I\{e_i^2 \leq e_{(\ell)}^2\} + o_p(1). \tag{98}$$

Now we may write (97) as

$$\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_i e_i \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right] \tag{99}$$

$$+\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_i e_i \cdot I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}.$$

Let us consider at first for any $j \in \{2, 3, ..., p\}$

$$\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_{ij} e_i \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right]. \tag{100}$$

For any $\varepsilon > 0$ we have

$$P\left(\frac{1}{\sqrt{n}}\left|\sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_{ij} e_i \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right]\right| > \varepsilon\right)$$

$$\leq \frac{1}{\varepsilon^2 n} I\!E\left\{\sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_{ij} e_i \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right]\right\}^2$$

$$= \frac{1}{\varepsilon^2 n} I\!E\left\{\sum_{i=1}^{n} Z_{ij} e_i \cdot \sum_{\ell=1}^{n} w_\ell^* \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right]\right\}^2. \tag{101}$$

Let us recall that the instruments Z_i's and the disturbances e_i's are independent and $I\!E Z_{ij} = 0$ for any $j \in \{2, 3, ..., p\}$. That is why for any $i, k = 1, 2, ..., i \neq k$ we have

$$I\!E\left\{Z_{ij} e_i \cdot \sum_{\ell=1}^{n} w_\ell^* \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right] \cdot Z_{kj} e_k \cdot \sum_{\ell=1}^{n} w_\ell^* \left[I\{e_k^2 \leq e_{(\ell)}^2\} - I\{e_k^2 \leq u_{1-\frac{\ell}{n}}^2\}\right]\right\}$$

$$= I\!E Z_{ij} \cdot I\!E Z_{kj} \times$$

$$\times I\!E\left\{e_i \cdot \sum_{\ell=1}^{n} w_\ell^* \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right] \cdot e_k \cdot \sum_{\ell=1}^{n} w_\ell^* \left[I\{e_k^2 \leq e_{(\ell)}^2\} - I\{e_k^2 \leq u_{1-\frac{\ell}{n}}^2\}\right]\right\} = 0.$$

It implies that (101) is equal to

$$\frac{1}{\varepsilon^2 n}\sum_{i=1}^{n} I\!E Z_{ij}^2 \cdot I\!E\left\{e_i \cdot \sum_{\ell=1}^{n} w_\ell^* \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right]\right\}^2$$

$$= \frac{1}{\varepsilon^2 n}\sum_{i=1}^{n} I\!E Z_{ij}^2 \cdot I\!E\left\{e_i^2 \cdot \sum_{\ell=1}^{n} w_\ell^* \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right] \times\right.$$

$$\times \sum_{k=1}^{n} w_k^* \cdot \left[I\{e_i^2 \leq e_{(k)}^2\} - I\{e_i^2 \leq u_{1-\frac{k}{n}}^2\} \right] \Bigg\}$$

$$\leq \frac{1}{\varepsilon^2 n} \sum_{i=1}^{n} I\!E Z_{ij}^2 \cdot I\!E \left\{ e_i^2 \cdot \sum_{\ell=1}^{n} w_\ell^* \cdot \left| I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right| \times \right.$$

$$\left. \times \sum_{k=1}^{n} w_k^* \cdot \left| I\{e_i^2 \leq e_{(k)}^2\} - I\{e_i^2 \leq u_{1-\frac{k}{n}}^2\} \right| \right\}. \quad (102)$$

Due to the fact that $\left| I\{e_i^2 \leq e_{(k)}^2\} - I\{e_i^2 \leq u_{1-\frac{k}{n}}^2\} \right| \leq 1$ and $\sum_{k=1}^{n} w_k^* = 1$, we have $\sum_{k=1}^{n} w_k^* \cdot \left| I\{e_i^2 \leq e_{(k)}^2\} - I\{e_i^2 \leq u_{1-\frac{k}{n}}^2\} \right| \leq 1$ and hence (102) is not larger than

$$\frac{1}{\varepsilon^2 n} \sum_{i=1}^{n} I\!E Z_{ij}^2 \cdot I\!E \left\{ e_i^2 \cdot \sum_{\ell=1}^{n} w_\ell^* \cdot \left| I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right| \right\}. \quad (103)$$

$$= \frac{1}{\varepsilon^2 n} \sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} I\!E Z_{ij}^2 \cdot I\!E \left\{ e_i^2 \cdot \left| I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right| \right\}$$

Then using Hőlder's inequality (for q' defined by $\frac{1}{q} + \frac{1}{q'} = 1$ where q is given in **C2**; notice please that $q > 1$ implies that $q' > 1$ as well) we find that (103) is not larger than

$$\frac{1}{\varepsilon^2 n} \sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} I\!E Z_{ij}^2 \cdot \left\{ I\!E \left| e_i \right|^{2q} \right\}^{\frac{1}{q}} \cdot \left\{ I\!E \left| I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right| \right\}^{\frac{1}{q'}}$$

$$= \frac{1}{\varepsilon^2} \sum_{\ell=1}^{n} w_\ell^* I\!E Z_{1j}^2 \cdot \left\{ I\!E \left| e_1 \right|^{2q} \right\}^{\frac{1}{q}} \cdot \left\{ I\!E \left| I\{e_1^2 \leq e_{(\ell)}^2\} - I\{e_1^2 \leq u_{1-\frac{\ell}{n}}^2\} \right| \right\}^{\frac{1}{q'}}. \quad (104)$$

Moreover, we have used

$$\left| I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right|^{q'} = \left| I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right|.$$

Utilizing Lemma 6, we find that there is $K < \infty$ and $n_1 \in N$ so that for $n > n_1$ we have $I\!E \left| I\{e_1^2 \leq e_{(\ell)}^2\} - I\{e_1^2 \leq u_{1-\frac{\ell}{n}}^2\} \right| < n^{-\frac{1}{2}} K$. It implies that (104) is bounded by

$$K^{\frac{1}{q'}} \varepsilon^{-2} n^{-\frac{1}{2q'}} \sum_{\ell=1}^{n} w_\ell^* I\!E Z_{1j}^2 \cdot \left\{ I\!E \left| e_1 \right|^{2q} \right\}^{\frac{1}{q}} = O(n^{-\frac{1}{2q'}}).$$

Now let us turn to (99) for $j = 1$, i.e. (remember that $Z_{i1} = 1$ for all i) we shall consider

$$\frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} e_i \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\} \right]. \quad (105)$$

We are going to show that it is small in probability.

In order to do it, let us recall once again that $r_{(\ell)}^2(\beta^0) = e_{(\ell)}^2$ and employing Lemma 6 let us find $n_1 \in N$ and $K^{(\varepsilon)} > 0$ such that for any $n > n_1$ and any $\ell = 1, 2, ..., n$ there is the interval $I_{1-\frac{\ell}{n},n}^{(\varepsilon)}$ so that

$$u_{1-\frac{\ell}{n}}^2 \in I_{1-\frac{\ell}{n},n}^{(\varepsilon)},$$

the set

$$B_n = \bigcap_{\ell=1,2,\ldots,n} \left\{ \omega \in \Omega \;:\; e^2_{(\ell)} \in I^{(\varepsilon)}_{1-\frac{\ell}{n},n} \right\}$$

has probability

$$P(B_n) > 1 - \frac{\varepsilon}{4}$$

and

$$I\!\!E \left[|e_i| \cdot I^{(\varepsilon)}_{1-\frac{\ell}{n},n} \right] \le n^{-\frac{1}{2}} K^{(\varepsilon)} \tag{106}$$

(see (53)). Then evidently for any $\omega \in B_n$

$$\left[e^2_{(\ell_n)}, u^2_{1-\frac{\ell}{n},n} \right]_{ord} \subset I^{(\varepsilon)}_{1-\frac{\ell}{n},n}. \tag{107}$$

Now, fixing $\varepsilon > 0$ and $\theta > 0$, let us find $\alpha_0 \in (0,1)$ so that $w(\alpha_0) - w(1) \le \frac{\varepsilon \cdot \theta}{8 \cdot K^{(\varepsilon)}}$ and $n_0 \in N$ so that $\frac{1}{n_0} < 1 - \alpha_0$. Recalling that we have put for any $\alpha \in (0,1)$ and $n \in N$ $\ell_n(\alpha) = [(1-\alpha)n]_{int}$, we have

$$\sum_{\ell=\ell_n(\alpha_0)}^{n} w_\ell = w(\alpha_0) - w(1) \le \frac{\varepsilon \cdot \theta}{8 \cdot K^{(\varepsilon)}}$$

and let us write (105) as

$$\frac{1}{\sqrt{n}} \sum_{\ell=1}^{\ell_n(\alpha_0)} w^*_\ell \sum_{i=1}^{n} e_i \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right] \tag{108}$$

$$+ \frac{1}{\sqrt{n}} \sum_{\ell=\ell_n(\alpha_0)}^{n} w^*_\ell \sum_{i=1}^{n} e_i \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right]. \tag{109}$$

Let us consider at first (109) and show that this is small in probability. We have

$$P\left(\frac{1}{\sqrt{n}} \left| \sum_{\ell=\ell_n(\alpha_0)}^{n} w^*_\ell \sum_{i=1}^{n} e_i \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right] \right| > \theta \right)$$

$$= P\left(\frac{1}{\sqrt{n}} \left| \sum_{\ell=\ell_n(\alpha_0)}^{n} w^*_\ell \sum_{i=1}^{n} e_i \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right] \cdot [I\{B_n\} + I\{B_n^c\}] \right| > \theta \right)$$

$$\le P\left(\frac{1}{\sqrt{n}} \left| \sum_{\ell=\ell_n(\alpha_0)}^{n} w^*_\ell \sum_{i=1}^{n} e_i \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right] \cdot I\{B_n\} \right| > \frac{\theta}{2} \right)$$

$$+ P\left(\frac{1}{\sqrt{n}} \left| \sum_{\ell=\ell_n(\alpha_0)}^{n} w^*_\ell \sum_{i=1}^{n} e_i \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right] \cdot I\{B_n^c\} \right| > \frac{\theta}{2} \right)$$

$$\le P\left(\frac{1}{\sqrt{n}} \left| \sum_{\ell=\ell_n(\alpha_0)}^{n} w^*_\ell \sum_{i=1}^{n} e_i \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right] \cdot I\{B_n\} \right| > \frac{\theta}{2} \right) + \frac{\varepsilon}{4}$$

$$\le P\left(\frac{1}{\sqrt{n}} \sum_{\ell=\ell_n(\alpha_0)}^{n} w^*_\ell \sum_{i=1}^{n} |e_i| \cdot \left| I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\} \right| \cdot I\{B_n\} > \frac{\theta}{2} \right) + \frac{\varepsilon}{4}.$$

Since $\left|I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right| = 1$ iff $e_i \in \left[e_{(\ell)}^2, u_{1-\frac{\ell}{n}}^2\right]_{ord}$, (107) implies that

$$P\left(\frac{1}{\sqrt{n}}\sum_{\ell=\ell_n(\alpha_0)}^{n} w_\ell^* \sum_{i=1}^{n} |e_i| \cdot \left|I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right| \cdot I\{B_n\} > \frac{\theta}{2}\right)$$

$$\leq P\left(\frac{1}{\sqrt{n}}\sum_{\ell=\ell_n(\alpha_0)}^{n} w_\ell^* \sum_{i=1}^{n} |e_i| \cdot I_{1-\frac{\ell}{n},n}^{(\varepsilon)} \cdot I\{B_n\} > \frac{\theta}{2}\right)$$

$$\leq I\!\!E\left\{\frac{2}{\theta\cdot\sqrt{n}}\sum_{\ell=\ell_n(\alpha_0)}^{n} w_\ell^* \sum_{i=1}^{n} |e_i| \cdot I_{1-\frac{\ell}{n},n}^{(\varepsilon)} \cdot I\{B_n\}\right\}$$

$$\leq \frac{2}{\theta\cdot\sqrt{n}}\sum_{\ell=\ell_n(\alpha_0)}^{n} w_\ell^* \sum_{i=1}^{n} I\!\!E\left[|e_i| \cdot I_{1-\frac{\ell}{n},n}^{(\varepsilon)}\right]$$

$$\leq \frac{2}{\theta\cdot n}\sum_{\ell=\ell_n(\alpha_0)}^{n} w_\ell^* \sum_{i=1}^{n} K^{(\varepsilon)} = \frac{2\cdot K^{(\varepsilon)}}{\theta}\sum_{\ell=\ell_n(\alpha_0)}^{n} w_\ell^* < \frac{\varepsilon}{4}.$$

Let us turn to (108), writing it as

$$\frac{1}{\sqrt{n}}\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n}\left[e_i - u_{1-\frac{\ell}{n}}\right] \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right] \tag{110}$$

$$+\frac{1}{\sqrt{n}}\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right] \tag{111}$$

$$+\frac{1}{\sqrt{n}}\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n}\left[e_i + u_{1-\frac{\ell}{n}}\right] \cdot I\{e_i < 0\} \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right] \tag{112}$$

$$-\frac{1}{\sqrt{n}}\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i < 0\} \cdot \left[I\{e_i^2 \leq e_{(\ell)}^2\} - I\{e_i^2 \leq u_{1-\frac{\ell}{n}}^2\}\right]. \tag{113}$$

We are going to make an idea at first about (110). Let us recall that we have put for any $\alpha \in (0,1)$ and $n \in N$ $\ell_n(\alpha) = [(1-\alpha)n]_{int}$ and hence for any $\alpha \in (\alpha_0, 1)$

$$1 - \frac{\ell_n(\alpha)}{n} = 1 - \frac{[(1-\alpha)n]_{int}}{n} \geq 1 - \frac{(1-\alpha)n}{n} = \alpha \geq \alpha_0$$

and hence for any $n \in N$ and $\ell = 1, 2, ..., \ell_n(\alpha_0)$

$$u_{1-\frac{\ell_n(\alpha)}{n}}^2 \leq u_{\alpha_0}^2. \tag{114}$$

We'll employ Lemmas 7 and 8 for $\delta = \alpha_0$, i. e. for $u_\delta^2 = u_{\alpha_0}^2$. For ε, we have fixed in previous, let us find $K_{\varepsilon,1} < \infty$ and $n_\varepsilon \in N$ such that for all $n > n_\varepsilon$, putting

$$B_n^{(1)} = \left\{\sup_{\alpha\in(\alpha_0,1)} \left|\sqrt{e_{(\ell_n(\alpha))}^2} - u_\alpha\right| < n^{-\frac{1}{2}} \cdot K_{\varepsilon,1}\right\} \tag{115}$$

and

$$B_n^{(2)} = \left\{ \sup_{\alpha\in(\alpha_0,1)} \left| e^2_{(\ell_n(\alpha))} - u^2_\alpha \right| < n^{-\frac{1}{2}} \cdot K_{\varepsilon,1} \right\},$$

the set $B_n = B_n^{(1)} \cap B_n^{(2)}$ has probability

$$P\left(B_n\right) > 1 - \frac{\varepsilon}{128}.$$

Further, utilizing Corollary 3 let us find $n_{\varepsilon,2} \in N$ and $K_{\varepsilon,2} < \infty$ such that for all $n > n_{\varepsilon,2}$ there is a set C_n such that $P(C_n) > 1 - \frac{\varepsilon}{128}$ and (see (66))

$$P\left(\left\{\left|I\left\{e_i^2 \le e^2_{(\ell_n(\alpha))}\right\} - I\left\{e_i^2 \le u^2_\alpha\right\}\right| = 1\right\} \cap I\{C_n\}\right) < n^{-\frac{1}{2}} \cdot K_{\varepsilon,2}. \qquad (116)$$

Let us put $K_\varepsilon = \max\{K_{\varepsilon,1}, K_{\varepsilon,2}\}$, $\quad n_\varepsilon = \max\{n_{\varepsilon,1}, n_{\varepsilon,1}, 128^2 \cdot \theta^{-2} \cdot \varepsilon^{-2} \cdot K_\varepsilon^4\}$, $D_n = B_n \cap C_n$ and consider $n > n_\varepsilon$ (for θ see the line under (107)). Then we have $P(D_n) > 1 - \frac{\varepsilon}{64}$ and due to (115) and (116) we can estimate (110)

$$P\left(\left\{\omega\in\Omega : \frac{1}{\sqrt{n}}\left|\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} \left[e_i - u_{1-\frac{\ell}{n}}\right] \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\}\right]\right| > \theta\right\}\right)$$

$$\le P\left(\left\{\omega\in\Omega : \frac{1}{\sqrt{n}}\left|\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} \left[e_i - u_{1-\frac{\ell}{n}}\right] \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\}\right]\right| > \frac{\theta}{2}\right\} \cap D_n\right)$$

$$+P\left(\left\{\omega\in\Omega : \frac{1}{\sqrt{n}}\left|\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} \left[e_i - u_{1-\frac{\ell}{n}}\right] \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\}\right]\right| > \frac{\theta}{2}\right\} \cap D_n^c\right)$$

$$\le \frac{2}{\theta\cdot n}\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} K_{(\varepsilon)} \cdot I\!E\left\{\left|I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\}\right|\right\} + \frac{\varepsilon}{64}$$

$$< 2\cdot\theta^{-1}\cdot n^{-\frac{1}{2}} \cdot K^2_{(\varepsilon)} \sum_{\ell=1}^{\ell_n} w_\ell^* + \frac{\varepsilon}{64} < \frac{\varepsilon}{16}.$$

Along the same lines we can treat (112). Let us turn now to (111). We have again

$$P\left(\left\{\omega\in\Omega : \frac{1}{\sqrt{n}}\left|\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\}\right]\right| > \theta\right\}\right)$$

$$\le P\left(\left\{\omega\in\Omega : \frac{1}{\sqrt{n}}\left|\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\}\right]\right| > \frac{\theta}{2}\right\} \cap D_n\right)$$

$$+P\left(\left\{\omega\in\Omega : \frac{1}{\sqrt{n}}\left|\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\}\right]\right| > \frac{\theta}{2}\right\} \cap D_n^c\right)$$

$$\le P\left(\left\{\omega\in\Omega : \frac{1}{\sqrt{n}}\left|\sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \le e^2_{(\ell)}\} - I\{e_i^2 \le u^2_{1-\frac{\ell}{n}}\}\right]\right| > \frac{\theta}{2}\right\} \cap D_n\right) + \frac{\varepsilon}{64}.$$

Now, employing Lemma 9, let us find $n_{\varepsilon,1} \in N$ and a finite constant $K^{(\varepsilon)}$ so that for any $n > n_\varepsilon$ there is a set A_n such that $P(A_n) > 1 - \frac{\varepsilon}{32}$ and

$$\sup_{\alpha \in (\theta,1)} \left| I\!\!E \left\{ I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e^2_{(\ell_n(\alpha))}\} - I\{e_i^2 \leq u^2_{1-\frac{\ell_n(\alpha)}{n}}\} \right] \cdot I\{A_n\} \right\} \right| < n^{-1} \cdot K^{(\varepsilon)}.$$

Further, finding $n_{\varepsilon,2} > \max \left\{ n_{\varepsilon,1}, \left[\frac{64}{\theta \cdot \varepsilon} u_{\alpha_0} \cdot K^{(\varepsilon)} \right]^2 \right\}$ and recalling that $n \in N$ and $\ell = 1, 2, ..., \ell_n(\alpha_0)$

$$u_{1-\frac{\ell_n(\alpha)}{n}} \leq u_{\alpha_0},$$

see (114), we have

$$P\left(\left\{ \omega \in \Omega : \frac{1}{\sqrt{n}} \left| \sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e^2_{(\ell)}\} - I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} \right] \right| > \frac{\theta}{2} \right\} \cap D_n \right)$$

$$\leq P\left(\left\{ \omega \in \Omega : \frac{1}{\sqrt{n}} \left| \sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e^2_{(\ell)}\} - I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} \right] \right| > \frac{\theta}{4} \right\} \cap D_n \cap A_n \right)$$

$$+P\left(\left\{ \omega \in \Omega : \frac{1}{\sqrt{n}} \left| \sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e^2_{(\ell)}\} - I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} \right] \right| > \frac{\theta}{4} \right\} \cap D_n \cap A_n^c \right)$$

$$\leq P\left(\left\{ \omega \in \Omega : \frac{1}{\sqrt{n}} \left| \sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e^2_{(\ell)}\} \right.\right.\right.\right.$$

$$\left.\left.\left.\left. - I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} \right] \right| > \frac{\theta}{4} \right\} \cap D_n \cap A_n \right) + \frac{\varepsilon}{32}$$

$$\leq \frac{4}{\theta\sqrt{n}} \sum_{\ell=1}^{\ell_n} w_\ell^* \sum_{i=1}^{n} u_{1-\frac{\ell}{n}} \cdot I\!\!E \left\{ I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e^2_{(\ell)}\} - I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} \right] \cdot I\{D_n\} \cdot I\{A_n\} \right\} + \frac{\varepsilon}{32}$$

$$\leq \frac{4}{\theta\sqrt{n}} \sum_{\ell=1}^{\ell_n} w_\ell^* u_{\alpha_0} \cdot \sum_{i=1}^{n} I\!\!E \left\{ I\{e_i > 0\} \cdot \left[I\{e_i^2 \leq e^2_{(\ell)}\} - I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} \right] \cdot I\{A_n\} \right\} + \frac{\varepsilon}{32}$$

$$\leq \frac{2}{\theta\sqrt{n}} \sum_{\ell=1}^{\ell_n} w_\ell^* u_{\alpha_0} \cdot \sum_{i=1}^{n} n^{-1} \cdot K^{(\varepsilon)} + \frac{\varepsilon}{32}$$

$$\leq \frac{2}{\theta\sqrt{n}} u_{\alpha_0} \cdot K^{(\varepsilon)} + \frac{\varepsilon}{32} \leq \frac{\varepsilon}{16}.$$

Along similar lines we can of course cope with (113). We conclude that (97) can be written as

$$\frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_i \left(Y_i - X_i'\beta^0 \right) I\{e_i^2 \leq e^2_{(\ell)}\} = \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_i e_i \cdot I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} + o_p(1)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i e_i \cdot \sum_{\ell=1}^{n} w_\ell^* \cdot I\{e_i^2 \leq u^2_{1-\frac{\ell}{n}}\} + o_p(1). \tag{117}$$

Now, recalling that $w_\ell^* = w\left(\frac{\ell-1}{n}\right) - w\left(\frac{\ell}{n}\right)$, we have

$$\sum_{\ell=1}^{n} w_\ell^* \cdot I\{e_i^2 \le u_{1-\frac{\ell}{n}}^2\}$$

$$= \sum_{\ell=\ell_n^{(i)}}^{n} \left[w\left(\frac{\ell-1}{n}\right) - w\left(\frac{\ell}{n}\right)\right] = w\left(\frac{\ell_n^{(i)}-1}{n}\right)$$

where $\ell_n^{(i)}$ is the smallest $\ell \in \{1,2,...,n\}$ such that $u^2_{1-\frac{\ell_n^{(i)}}{n}} > e_i^2$. Due to the fact that (remember the continuity of $F_{\beta^0}(z)$ from the left)

$$F_{\beta^0}\left(u_{1-\frac{\ell_n^{(i)}-1}{n}}\right) = 1 - \frac{\ell_n^{(i)}}{n} \ge F_{\beta^0}(|e_i|) \ge 1 - \frac{\ell_n^{(i)}}{n},$$

we have due to monotonicity of the weight function w

$$w\left(1 - \frac{\ell_n^{(i)}-1}{n}\right) \le w\left(F_{\beta^0}(|e_i|)\right) \le w\left(1 - \frac{\ell_n^{(i)}}{n}\right)$$

and finally (due to the fact that the derivative $w'(u)$ is bounded)

$$\sum_{\ell=\ell_n^{(i)}}^{n} \left[w\left(\frac{\ell-1}{n}\right) - w\left(\frac{\ell}{n}\right)\right] = w\left(F_{\beta^0}(|e_i|)\right) + o(\frac{1}{n}).$$

Hence we have

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i e_i \cdot \sum_{\ell=1}^{n} w_\ell^* \cdot I\{e_i^2 \le u_{1-\frac{\ell}{n}}^2\} + o_p(1) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} w\left(F_{\beta^0}(|e_i|)\right) Z_i e_i + o_p(1). \tag{118}$$

Returning to (98), we may write the first term as

$$\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n} w_\ell^* \sum_{i=1}^{n} Z_i X_i' \left(\hat\beta^{(IWV,n,w)} - \beta^0\right) \cdot I\{e_i^2 \le e_{(\ell)}^2\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} Z_i X_i' \cdot \left[\sum_{\ell=1}^{n} w_\ell^* I\{e_i^2 \le e_{(\ell)}^2\}\right] \cdot \left\{\sqrt{n}\left(\hat\beta^{(IWV,n,w)} - \beta^0\right)\right\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} w\left(F_{\beta^0}^{(n)}(|e_i|)\right) Z_i X_i' \cdot \left\{\sqrt{n}\left(\hat\beta^{(IWV,n,w)} - \beta^0\right)\right\} \tag{119}$$

where we have used (48). Further we can write the last expression as

$$\frac{1}{n}\sum_{i=1}^{n} \left\{ w\left(F_{\beta^0}^{(n)}(|e_i|)\right) - w\left(F_{\beta^0}(|e_i|)\right)\right\} \cdot Z_i X_i' \cdot \left\{\sqrt{n}\left(\hat\beta^{(IWV,n,w)} - \beta^0\right)\right\} \tag{120}$$

$$+\frac{1}{n}\sum_{i=1}^{n} w\left(F_{\beta^0}(|e_i|)\right) \cdot Z_i X_i' \cdot \left\{\sqrt{n}\left(\hat\beta^{(IWV,n,w)} - \beta^0\right)\right\} \tag{121}$$

Now employing Assertion A.5 and the assumption **NC2**, writing (120) as

$$\left\| n^{-\frac{3}{2}} \sum_{i=1}^{n} \sqrt{n} \left\{ w\left(F_{\beta^0}^{(n)}(|e_i|)\right) - w\left(F_{\beta^0}(|e_i|)\right) \right\} \cdot Z_i X_i' \cdot \left\{ \sqrt{n}\left(\hat{\beta}^{(IWV,n,w)} - \beta^0\right)\right\} \right\|$$

$$\leq n^{-\frac{1}{2}} \sup_{r \in R} \sqrt{n} \left| w\left(F_{\beta^0}^{(n)}(|r|)\right) - w\left(F_{\beta^0}(|r|)\right)\right| \cdot \frac{1}{n} \sum_{i=1}^{n} \|Z_i\| \cdot \|X_i\| \cdot \left\{ \sqrt{n} \left| \hat{\beta}^{(IWV,n,w)} - \beta^0 \right| \right\},$$

and taking into account $\sqrt{n}$-consistency of $\hat{\beta}^{(IWV,n,w)}$, we find that (120) is $\mathcal{O}_p(n^{-\frac{1}{2}})$. Now, employing (118), (119) and (121), we conclude the proof. □

Corollary 4 *Let the conditions* **C1***,* **C2***,* **C3***,* **C4***,* **NC1***,* **NC2***,* **AC1** *and* **AC2** *hold and let* $Q = I\!\!E\left\{w(F_{\beta^0}(|e|))Z_1X_1'\right\}$*. Then*

$$\sqrt{n}\left(\hat{\beta}^{(IWV,n,w)} - \beta^0\right) = Q^{-1} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w\left(F_{\beta^0}(|e_i|)\right) \cdot Z_i e_i + o_p(1). \tag{122}$$

Proof follows immediately from the fact that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} w\left(F_{\beta^0}(|e_i|)\right) Z_i X_i' = Q \qquad \text{a.s.}$$

and Q is positive definite and hence regular. □

Appendix

Lemma A.1 *(Štěpán (1987), page 420, VII.2.8) Let a and b be positive numbers. Further let ξ be a random variable such that $P(\xi = -a) = \pi$ and $P(\xi = b) = 1 - \pi$ (for a $\pi \in (0,1)$) and $I\!\!E\xi = 0$. Moreover let τ be the time for the Wiener process $W(s)$ to exit the interval $(-a, b)$. Then*

$$\xi =_{\mathcal{D}} W(\tau)$$

where "$=_{\mathcal{D}}$" denotes the equality of distributions of the corresponding random variables. Moreover, $I\!\!E\tau = a \cdot b = var\,\xi$.

Remark A.1 *Since the book by Štěpán (1987) is in Czech language we refer also to Breiman (1968) where however this simple assertion is not isolated. Nevertheless, the assertion can be found directly in the first lines of the proof of Proposition 13.7 (page 277) of Breiman's book. (See also Theorem 13.6 on the page 276.)*

Lemma A.2 *(Štěpán (1987), page 409, VII.1.6) Let a and b be positive numbers. Then*

$$P\left(\max_{0\leq t\leq b} |W(t)| > a\right) \leq 2 \cdot P\left(|W(b)| > a\right).$$

Remark A.2 *Since, as we have already said, the book by Štěpán (1987) is in Czech language we refer again also to Breiman (1968) where however seemingly only weaker assertion can be found, see Proposition 12.20 (page 258) of Breiman's book.*

Lemma A.3 *Let for some* $p \in N$, $\left\{\mathcal{V}^{(n)}\right\}_{n=1}^{\infty}$, $\mathcal{V}^{(n)} = \left\{v_{ij}^{(n)}\right\}_{i=1,2,...,p}^{j=1,2,...,p}$ *be a sequence of* $(p \times p)$ *matrixes such that for* $i = 1, 2, ..., p$ *and* $j = 1, 2, ..., p$

$$\lim_{n\to\infty} v_{ij}^{(n)} = q_{ij} \qquad \text{in probability} \tag{A.1}$$

where $Q = \{q_{ij}\}_{i=1,2,...,p}^{j=1,2,...,p}$ *is a fixed nonrandom regular matrix. Moreover, let* $\left\{\theta^{(n)}\right\}_{n=1}^{\infty}$ *be a sequence of* p*–dimensional random vectors such that*

$$\exists\ (\varepsilon > 0)\ \forall\ (K > 0)\ \limsup_{n\to\infty} P\left(\|\theta^{(n)}\| > K\right) > \varepsilon.$$

Then

$$\exists\ (\ \delta > 0) \quad \forall\ (H > 0)$$

so that

$$\limsup_{n\to\infty}\ P\left(\left\|\mathcal{V}^{(n)}\theta^{(n)}\right\| > H\right) > \delta.$$

Proof: Due to (A.1) the matrix $\mathcal{V}^{(n)}$ is regular in probability. Let then $0 < \lambda_{1n} < \lambda_{2n} < ... < \lambda_{pn}$ and $z_{1n}, z_{2n}, ..., z_{pn}$ be eigenvalues and corresponding eigenvectors (selected to be mutually orthogonal) of the matrix $[\mathcal{V}^{(n)}]^T\mathcal{V}^{(n)}$. Let us write $\theta^{(n)} = \sum_{j=1}^{p} a_{jn}z_{jn}$ (for an appropriate vector $a_n = (a_{1n}, a_{1n}, ..., a_{pn})^T$). Then we have

$$\left\|\mathcal{V}^{(n)}\theta^{(n)}\right\|^2 = \sum_{j=1}^{p}[a_{jn}]^2\lambda_{jn}\|z_{jn}\|^2 \leq \lambda_{1n}\|\theta^{(n)}\|. \tag{A.2}$$

Moreover, denoting λ_1 the smallest eigenvalue of the matrix Q^TQ, we have $\lambda_{1n} \to \lambda_1$ in probability as $n \to \infty$. The assertion of the lemma then follows from (A.2). □

Assertion A.1 *Let* $\{e_i\}_{i=1}^{\infty}$ $(e_i \in R)$ *be a sequence of independent and identically distributed random variables with absolutely continuous distribution function* $F(z)$*. Then for any* $n \in N$ *and any* $i, \ell = 1, 2, ..., n$ *we have*

$$P(r_i^2(\beta^0) = r_{(\ell)}^2(\beta^0)) = \frac{1}{n}.$$

Proof. The proof can be found in Čížek (1996). Since it is not easy available, let us give it (moreover, it's short). First of all, let us recall that for any $i \neq j, i, j = 1, 2, ..., n$ and $n \in N$

$$P(r_i^2(\beta^0) = r_j^2(\beta^0)) = 0.$$

Due to the fact that the random variables e_i's are i.i.d., we have for all pairs $i, j = 1, 2, ..., n$

$$P(r_i^2(\beta^0) = r_{(\ell)}^2(\beta^0)) = P(r_j^2(\beta^0) = r_{(\ell)}^2(\beta^0))$$

and

$$\sum_{i=1}^{n} P(r_i^2(\beta^0) = r_{(\ell)}^2(\beta^0)) = 1.$$

That concludes the proof. □

Assertion A.2 *(Rao (1973), 1e.7) Stirling's formula is given as*

$$n! = \sqrt{2\pi n} \cdot n^n \cdot \exp\{-n\} \cdot \exp\{\frac{\lambda(n)}{12}\}$$

where $(n + \frac{1}{2})^{-1} < \lambda(n) < n^{-1}$.

Lemma A.4 *(Rao (1973) 6f.2, Theorem II) Let* $\{X_n\}_{n=1}^{\infty}$ *be the sequence of i.i.d. r.v.'s distributed according to d.f.* $F(x)$ *with a continuous density* $f(x)$*. Moreover, let for* $\alpha \in (0,1)$ *the quantile* u_α *(defined by* $F(u_\alpha) = 1 - \alpha$*) be given uniquely and* $f(u_\alpha) > 0$*. Finally, put* $\hat{u}_\alpha = X_{(\ell_n(\alpha))}$ *where* $\ell_n(\alpha) = [(1-\alpha)n]_{int}$*. Then*

$$\sqrt{n}\,(\hat{u}_\alpha - u_\alpha) \underset{D}{\longrightarrow} \mathcal{N}\left(0, \frac{\alpha(1-\alpha)}{f^2(u_\alpha)}\right).$$

Proof: Although the proof can be found in famous Rao's book, we sketch it because we shall need some relations from it. Evidently

$$F_{\hat{u}_\alpha}(x) = P(\hat{u}_\alpha < x) = P(\{\omega \in \Omega : \#\{X_i(\omega) < x, i = 1, 2, ..., n\} \geq \ell_n(\alpha)\})$$
$$= \sum_{\ell=\ell_n(\alpha)}^{n} \frac{n!}{\ell!(n-\ell)!} F^\ell(x)\,[1 - F(x)]^{n-\ell} \tag{A.3}$$

and hence the density of $\hat{u}_\alpha$ reads as follows

$$\frac{n!}{(\ell_n(\alpha)-1)!(n-\ell_n(\alpha))!}\, F^{\ell_n(\alpha)-1}(x)\,[1-F(x)]^{n-\ell_n(\alpha)}\, f(x). \tag{A.4}$$

Putting $y = F(x)$ (with, of course $\mathrm{d}y = f(x)\mathrm{d}x$), we obtain the density of r.v. $\hat{\xi} = F(\hat{u}_\alpha)$ as

$$\frac{n!}{(\ell_n(\alpha)-1)!(n-\ell_n(\alpha))!}\; y^{\ell_n(\alpha)-1}\,[1-y]^{n-\ell_n(\alpha)}$$

and employing the transformation

$$z = \frac{\sqrt{n}(y-\alpha)}{\sqrt{\alpha(1-\alpha)}}, \qquad \text{i. e.} \quad y = \alpha\left(1 + z\sqrt{\frac{1-\alpha}{n\alpha}}\right),$$

we arrive at the expression for density of the r.v. $\hat{\zeta} = \frac{\sqrt{n}(\hat{\xi}-\alpha)}{\sqrt{\alpha(1-\alpha)}}$

$$\frac{1}{\sqrt{n}} \cdot \frac{n!}{(\ell_n(\alpha)-1)!(n-\ell_n(\alpha))!}\; \alpha^{\ell_n(\alpha)-\frac{1}{2}}\,[1-\alpha]^{n-\ell_n(\alpha)+\frac{1}{2}} \times \tag{A.5}$$

$$\times \left(1 + z\sqrt{\frac{1-\alpha}{n\alpha}}\right)^{\ell_n(\alpha)-1} \left(1 - z\sqrt{\frac{\alpha}{n(1-\alpha)}}\right)^{n-\ell_n(\alpha)}. \tag{A.6}$$

Employing Stirling's formula we find that logarithm of (A.5) is bounded (from above and from below) by

$$-\frac{1}{2}\log(2\pi) \pm \eta(n) \tag{A.7}$$

where $\eta(n) = \lambda(n) - \lambda(\ell_n(\alpha) - 1) - \lambda(n - \ell_n(\alpha))$ (for $\lambda(n)$ see Assertion A.2). The logarithm of (A.6) is of course given by

$$(\ell_n(\alpha) - 1) \cdot \log\left(1 + z\sqrt{\frac{1-\alpha}{n\alpha}}\right) + (n - \ell_n(\alpha)) \cdot \log\left(1 - z\sqrt{\frac{\alpha}{n(1-\alpha)}}\right)$$

and hence utilizing Taylor's expansion for logarithm we find that it is equal to

$$-\frac{z^2}{2} + z^3 \cdot \mathcal{O}(n^{-\frac{1}{2}}). \tag{A.8}$$

Applying now Scheffe's theorem (see e. g. Rao (1973), 2c.4, Theorem XV), we conclude that r. v. $\hat{\zeta} = \frac{\sqrt{n}(\hat{\xi}-\alpha)}{\sqrt{\alpha(1-\alpha)}}$ converges in distribution to the standard normal one. Using the inverse transformation $\hat{u}_\alpha = F^{-1}(\hat{\xi})$, we conclude the proof. □

Corollary A.1 *Let the assumption of the previous lemma hold and $\alpha_0 \in (0,1)$. Then for any $\alpha \in (\alpha_0, 1)$ the density of $\hat{u}_\alpha = X_{(\ell_n(\alpha))}$ is given by*

$$h_{n,\alpha}(u) = h^*_{n,\alpha}(u) + \varrho_{n,\alpha}(u) \tag{A.9}$$

where $h^*_{n,\alpha}(r)$ is a density which is symmetric around u_α and for any finite K we have

$$\sup_{\alpha\in(\alpha_0,1)} \sup_{|u|\le n^{-\frac{1}{2}}K} |\varrho_{n,\alpha}(u)| = \mathcal{O}(n^{-\frac{1}{2}}). \tag{A.10}$$

Proof: Let us recall that the density given in (A.5) and (A.6) is the density of $\hat{\zeta} = \frac{\sqrt{n}(\hat{\xi}-\alpha)}{\sqrt{\alpha(1-\alpha)}}$. Then the assertion of corollary follows from (A.7) and (A.8). □

Lemma A.5 *Under* **Conditions C1** *we have*

$$\sup_{v\in \mathbb{R}^+,\ \beta\in \mathbb{R}^p} \sqrt{n}\left|F^{(n)}_\beta(v) - F_\beta(v)\right| = \mathcal{O}_p(1). \tag{A.11}$$

For the **proof** of lemma see Víšek (2006a), Collorary 14.

Definition A.1 *The stochastic process $V = (V(s),\ s \in S) \subset R^p, S \subset R^q,\ p, q \in N$, is called separable if there is a countable dense subset $T \subset S$ (i.e. T is countable and dense in S).*

Lemma A.6 *(Štěpán (1987), page 85, I.10.4) Let $V = (V(s),\ s \in S) \subset R^\ell,\ \ell \in N$ be a separable stochastic process defined on the probability space $(\Omega, \mathcal{A}, P)$. Moreover, let $G \subset S$ be open and denote by $k(G)$ the set of all finite subsets of G. Then for any close set $K \subset R^p$ we have*

$$\{\omega \in \Omega : V(s) \in K, s \in G\} \in \mathcal{A}$$

and

$$P\left(\{\omega \in \Omega : V(s) \in K, s \in G\}\right) = \inf_{J\in k(G)} P\left(\{\omega \in \Omega : V(s) \in K, s \in J\}\right).$$

Proof: Since the book by Štěpán is in Czech language and the proof is short, we will give it. Let T be countable dense subset of S. Then we have

$$\{\omega \in \Omega : V(s) \in K, s \in G\} = \{\omega \in \Omega : V(s) \in K, s \in G \cap T\}$$

and

$$P\left(\{\omega \in \Omega : V(s) \in K, s \in G\}\right) \leq \inf_{J \in k(G)} P\left(\{\omega \in \Omega : V(s) \in K, s \in J\}\right)$$

$$\leq \inf_{J \in k(G \cap S)} P\left(\{\omega \in \Omega : V(s) \in K, s \in J\}\right) = P\left(\{\omega \in \Omega : V(s) \in K, s \in G \cap S\}\right)$$

$$= P\left(\{\omega \in \Omega : V(s) \in K, s \in G\}\right). \qquad \square$$

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