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DIPLOMA THESIS

Voting Paradoxes

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Affirmation

I declare that I have elaborated this thesis individualy and have used but mentioned sources and literature.

13/05/2004 Prague Signature:

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Introduction

This thesis deals with paradoxes of power - situations that contradict intuitive expectations about the properties of voting power of different members in committees.

While the pioneers in the field, Marquis de Condorcet (1743-1794) and Jean Charles de Borda (1733-1799), showed a few paradoxical situations peculiar to certain voting procedures more than two centuries ago, power indices - concepts of measuring a committee members' voting power - were introduced much later, in the second half of the last century.

Although 'power' is intuitively well understood, formal definitions (e. g. Allingham, 1975) do not lead and cannot lead to the unique possible power measure. The starting point of power considerations is a simple model of a committee (called also a weighted voting body) described by distribution of votes among its members and by the quota (minimal number of votes necessary to pass a decision). Measuring power means evaluation of probability that a member of a committee will be in a situation that his vote is 'decisive' for the outcome of voting. However, definition of a concept 'to be decisive' is not unique and differs by different authors.

It is generally accepted that simple proportion of votes in a voting body is not a good proxy for the power measure. By developing initial work of Penrose (1946), Banzhaf and Coleman (1965, 1971) came to one basic concept of a power index (Penrose – Banzhaf – Coleman index) based on the concept of swing (ability to change voting coalition from the winning one to the losing one).

Shapley and Shubic (1954) introduced the concept of power measure on the basis of bargaining considerations introducing so called pivotal situation (position in the consecutive process of forming the voting coalition when a member by decision to join or not to join it decides whether it is winning or losing). Probability to be in a pivotal situation defines the Shapley – Shubic power index.

Yet, many more concepts exist (See e. g. Johnston (1978), Holler and Packel (1978, 1983), and Deegan and Packel (1978).), however, these are but merely derivatives of the two mentioned at first.

Distinguishing of 'the right' measure (if there is any) is strenuous work that hasn't been positively solved so far. However, it's of crucial importance. Analyses of a committee members' power are needful especially when weighted voting is involved. Such institutions are e. g. IMF, or the Council of Ministers as the most important decision – making body of the European Union. Also in any business corporation important decisions are voted upon with shares as votes.

How problematic the question is can be seen from US Courts' decisions in 1960s and 1970s that ruled the possibility of using weighted voting out in various types of local elections in lieu of reapportionment of (unequal) districts every decade.System of binding axioms on power indices has been introduced. As a result of local monotonicity no more that two already mentioned basic measures are acceptable.

Formal distinction of these two indices can be achieved by usage of additivity concept (Nurmi (1987)) but as other milestone in theory we consider global monotonicity (Turnovec). By global monotonicity Shapley – Shubic index is the best from axiomatic point of view. On the other hand, it is founded on bargaining assumption that may not hold. As a result it, to be precise, shouldn't be used for secrete voting where it cannot be verified (Felsenthal and Machover (2001)). If the assumption isn't satisfied, the index is biased in its outcome (Holler (2002), Loužek (2003)). Banzhaf – Coleman index, on the other hand cannot be used for comparing the powers of the same voter under different decision rules.

Even from this brief introduction into power indices it's clear one should pay attention to pre-requirements, indices' assumptions and strains. Having seen the fact that majority of analyses undertaken with help of power indices neglect this we have decided to have a closer look at indices' theory. Yet, most of it is binary, which means only two equiprobable decisions of a voter are allowed - 'yes' and 'no'.

Our hypotheses are:

1) Not all the voting paradoxes peculiar to power indices have been so far found. We should detect them and take into account when deciding about 'the best index'.

2) The fact that binary power indices are used for analyses of voting bodies; where abstention is allowed and often occurs; isn't theoretically justified. As a first step abstention and its outcome should be analysed.

3) Binary power indices are too simplifying. Their extension for abstention is desirable.

The work consists of three parts. The first one deals with committee systems and measures of power. Although the reader is likely to be well acquainted with the topic, for the sake of completeness, the most important definitions and theorems together with the most commonly used assumptions, their acceptability, importance and strains are discussed in section 1.1.

The following section explains normative nature of any power index, shows its inevitability and traces back to the pioneering concepts of Penrose (1946) and Shapley and Shubic (1954). Description of their work together with extension by Banzhaf (1965) is followed by bringing out that there have not been any other concepts brought about except merely derivatives of these. Here we also get to know restrictions on use of power measures implied by their nature.

Basic axioms of power measures are discussed in 1.3.1.Binding concepts leading to uniqueness of power index: additivity (Nurmi (1987)) and global monotonicity (Turnovec) are discussed in 1.3.2. Definition of global monotonicity is given equivalent reinterpretation and the idea of bargaining hidden behind mentioned.

Section 1.4 brings formal shape of the most commonly used power measures. It is shown that Shapley – Shubic relevance is based on bargaining assumption. If this didn't hold the index would be biased by its overvaluation of more membered coalitions.

Apart from this rather descriptive nature of the part we also deliver basic concept of simple proportionality and proof of its implausibility due to its not satisfying dummy member axiom (a player that cannot benefit any voting configuration by joining it should be assigned no value of power).

Section 1.5 opens possibilities of power indices' extensions. 1.5.1 deals with probabilistic approach to the distribution of power where the most important results were achieved by Straffin (1998), 1.5.2 deals with spatial models where basis were sat by Banks (1995), McKelvey and Schofield (1986) and Saari (1997). Extension for abstention is deeply dwelled into in parts 2 and 3.

Second part is devoted to voting paradoxes. Section 2.1 classifies them with emphasise onto those we need later, especially the paradoxical act of voting that tries to answer a crucial question why so many - usually most - eligible people vote even though the effect of their vote is very marginal and they incur real costs in voting.

As to the authors of the paradoxes or their solutions, let us mention at least Kemeny (1959), Gehrlein (1997), Ostrogorski (1903), Anscombe (1976), Brams, Kilgour and Zwicker (1998), Cohen and Nagel (1934), Simpson (1951), Schwartz (1995) and Nurmi.

Binary paradoxes of power are presented in part 2.2. Among them the paradox of redistribution implying a committee members' powers are interdependent (2.2.1).

Paradox of weighted voting discussed in 2.2.2 explains the concept of local monotonicity. Section 2.2.3 comes up with the paradoxes of threshold and of new members, while 2.2.4 with the paradoxes of co - operation including newly introduced Paradox of powerful sinners.

In extensive section 2.3 we, departing from the concept of general monotonicity (2.3.1), explicitly find a new voting paradox - the Paradox of abstention (subsection 2.3.2) describing a situation when a committee member's power increases despite his abstaining from voting with a certain part of his votes. It is of high importance for our second hypothesis and launches a new type of so far undocumented monotonicity problem.

In the same section we come to the result of abstention paradox being index specific (cannot occur for SSI) and detect bounds on committee member's number allowing the paradox's occurrence.

Sections 2.3.3 and 2.3.4 deal with the impact of Abstention paradox's existence on general monotonicity. We prove that the original non-normalised Banzhaf – Coleman index due to the existence of Abstention paradox fails to satisfy general monotonicity in far less cases.

It's known that Banzhaf - Coleman index and Shapley - Shubic index generally indicate different power values but: ‘In particular, in single-chamber legislatures or similar decision-making bodies using larger than simple majority rules, the indices give fairly similar results. This is, however, not necessarily the case in multi-chamber bodies.’[1]. What we proved is no less than the difference between these two indices is even smallish.

Thus, while general monotonicity looses part of its importance (necessary for distinction between the indices) we can see as a result of the proof that whenever non – normalised Banzhaf – Coleman index used, the results aren't a priory powers but the lower bound of what non – normalised Banzhaf – Coleman index claims to calculate.

Part 3 investigates our last hypothesis. Despite the fact that Abstention paradox itself doesn't justify departing from bipartition, possibility of describing situations when abstention is allowed by conventional power indices is limited to Shapley – Shubic index and non-existence of rational a priory abstention (section. 3.1).

We into details explain for committees with either quorum calculated from present members or from all, how's requirement of democratic elections violated. Yet, not only is abstention discriminated but as we proved there are situations when's passive abstention (a voter doesn’t come to take a part in a decision – making process) individually superior to active one (a voter is present and votes for ‘abstain’) and that's discrimination beyond the pale.

Based on analyses of the way abstention is commonly treated, section 3.2 shows necessity of institutional change and offers a proposal.

Section 3.3 comes with modelling of power indices extended for abstention. In 3.3.1 we get to know up – to – date existing models (Felsenthal&Machover’s and Braham&Steffen’s). We analyse their strains of unrealistic probabilities of abstention and no distinction between active and passive abstention. In 3.3.2 we show that due to the paradoxical act of voting passive abstention can occur and thus should be considered.

In the following section we introduce our own power index based on statistically backed assumption that a priory active abstention is equally likely to occur for every

[1] Nurmi (1998), p. 174

member of a coalition with a probability given by endogenous parameter *p*. The obtained index takes formal form of binomial distribution of respective subcoalitions derived from a committee analysed. It is, in fact, general version of known power indices and takes their form for parameter *p* equalling zero.

Section 3.3.4 shows difficulties we face when detecting value of parameter *p* together with tools we can use. About its quantification we are able to say enough to raise a hypothesis that binary indices of power do not give relevant and stable information for committees with a few tens of members or more.

The last section is an application of our index on data from Parliament of the Czech Republic leading to the conclusion that difference between binary indices and indices with tripartition may vary a lot with a committee structure.

1. Committee Systems and Measures of Power

1.1 Fundamental Concepts

We use the concept of power by Allingham: '***power** is interpreted as ability to influence outcomes - irrespective of the desirability of these; the concept of power should therefore be independent of any concept of utility*'.[2] And to be more precise, only one narrow feature of power - voting power in democratic society represented by a committee consisting of full members disposing with a given number of votes (or weight allocation) ω.

Voting power we define as the chance that a given member's vote will be crucial to the decision voted upon. Voting rules take on a form of simple or qualified majority.

A power index represents: '*a reasonable expectation of the share of decisional power among the various members of a committee, given by ability to contribute to formation of winning voting configurations. We shall denote by $\pi_i(\gamma,\omega)$ the share of power that the index π grants to the i-th member of a committee with weight allocation ω and quota γ. Such a share is called a power index of the i-th member.*'[3]

To get **a normalized power index**, numbers obtained for individual players are divided by their sum (if they do not sum up to unity).

Formally, a power index π counts one if it is of the form

$\pi_j(v) = \dfrac{\sum_{j\in S} c_v(j,S)}{\sum\limits_{k=1}^{n}\sum_{k\in T} c_v(k,T)}$, where coefficients $c_v(k,T)$ are nonnegative for all coalitions and fulfil the condition $T \notin D(k,v) \Rightarrow c_v(k,T) = 0$

If a power index is calculated irrespective of any further information, individual preferences or capacities, we call it **a priori**, unconstrained or abstract **power index**.

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[2] Allingham (1975), p. 293 in: Holler (2003), p. 39
[3] Turnovec (2003), p. 137

A priori voting power indices abstract from the preferences of the decision-makers unlike constrained[4] or empirical power indices (if real-world data incorporated). We can also recognize posterior (post-electoral or actual voting power). Thus a priory power is power that a member of a committee derives exclusively from the decision rule itself and is power in a constitutional sense.

The crucial question is, whether we should (or can) incorporate or alternate the given index we use with additional information we have about a certain decision body. The more the issue we analyze is of constitutional character, the most should the index ignore such factors, which are extraneous to the decision rule itself (or in other words omit up-to date information for the sake of its general properties). The most the answer is no. And that's why we should take a closer look at a priory power indices.

A different explanation of the same can be done the following way: a priory power measures are unrealistic in their assumptions about voters' behaviour, the decision making process, etc.[5] but all additional information is time dependent. If a long term is 'long enough'[6], a priory power is a very good estimator of long term actual power. Thus the main purpose of measuring a priory voting power is rather prescriptive than descriptive.

As we already mentioned, we need power indices for analyses of decision-making bodies. Such a body is usually called committee and is: *'…a set of individuals forming a joint opinion regarding the course of action (policy) to be taken, candidate to be elected, a statement to be issued or a body of individuals to be nominated.'*[7]

1.2 Two Distinct Concepts of Power

[4] For interesting work on constrained power indices see Dostálová (2003) who analyses the Czech chamber of deputies under the set of (in political sciences generally recognized) assumptions of different vulnerability to coalition formation according to the parties' left – right spectra, size of the prospective coalition etc. To the size of coalitions we'll refer in part 1.4.5.
[5] See for example Tsebelis (1999), or Moberg (2002)
[6] Compare with Turnovec (1997) in: Control and Cybernetics (1997)
[7] Nurmi (2003), p. 47

At closer inspection one realizes that the definitions from section 1.1 aren't sufficient for measuring a priory power. This is due to the normative nature of power indices. Why it is on one side inevitable and on the other the indices form a useful tool of public choice, we'll see in this section.

Comprehension of the pioneering works from the field of power measures prevents us from the most common fallacies. These mistakes, usually stemming from undesirable misrepresentation of power indices are so often, and the discussion on relevance of any a priory power measures is so lively that we feel an obligation of contributing by a brief summary of key pros and cons of the indices.

As far as we know, the first study of voting power on scientific basis was undertaken by Penrose (1946).[8] Under the assumption that all voters vote independently and at random he defines a voter's probability of success. For *r* standing for the proportion of all possible divisions of the assembly where a member *a* is successful, he gets $r \in \left\langle \frac{1}{2};1 \right\rangle$, where obviously r = 1/2 for a dummy. Such an interval caused by success not excluding luck isn't convenient. Therefore Penrose defined $\varphi_a = 2r_a - 1$ as a measure of a's voting power.[9] Equivalent interpretation of Penrose's φ_a is the probability that a given voter can reverse the outcome by reversing his vote (Banzhaf, 1965).

There are only two alternative approaches how to tackle with a priory voting power and a handful of their derivatives (see part 1.4). The basis of the other concept; an alternative to Penrose; the quantity known as Shapley value (Shapley and Shubik, 1954); is derived from the mathematical theory of cooperative games with transferable utility. It is accepted by many game – theorists as a prior probabilistic estimate of the payoff that the player can expect, on the average.

The basic strain of Shapley and Shubik's concept is the fact that there is no compelling and realistic general theory of bargaining for cooperative games with more than

[8] There had been some attempts even before, especially by Luther Martin, a Maryland delegate to the 1787 Constitutional Convention held in Philadelphia. But these attempts hadn't been systematic ones.
[9] According to Penrose a voter's power is equal to the probability of that voter being in a position to decide the outcome of a division.

two players. Yet, the assumption of bargaining is crucial for the Shapley value. For secret voting, the concept of Shapley, strictly speaking, shouldn't be applied at all, because there's no way of verifying if the players voted according to the bargaining process that should have preceded.

Penrose's and Shapley's Concepts Compared

The term I-power; introduced by Felsenthal and Machover; is used for voting power conceived of as a voter's potential influence over the outcome of divisions of the decision-making body, which is the same what Penrose says. The informational vacuum for a priory I-power implies that we have to - by the probabilistic Principle of Insufficient Reason (PIR) - assign equal prior probability to all possible divisions of the assembly.[10]

Incoherence of PIR for infinite probability spaces isn't of any relevance for a priory indices based on I-power since there is always the assumption of finite probability space with finitely many clearly distinguished indivisible 'atomic' events (2^n for Penrose with an assembly of n voters).

Contrary: *'By P-power we mean voting power conceived of as a voter's expected relative share in a fixed price available to the winning coalition under a decision rule, seen in the guise of a simple TU cooperative game.'*[11] This is exactly what Shapley value describes.

The contribution of the authors is of course not in the new labelling but in pointing out that while Penrose's index is absolute, the one by Shapley is relative. While under Penrose's whether a voter would vote for or against depends crucially on the specific bill and the voter's preferences, under Shapley's a voter with greater power has greater incentive and opportunity to become a member of a winning coalition.

1.3 The Axioms of Power Indices

[10] PIR is also known as the Principle of Indifference and is a special case of the Principal of Maximal Information Entropy. The Principle of Insufficient Reason goes back to Bernoulli (1654-1705) and for its critique see for example Keynes (1921), Urbach (1993), or Albert (2003).
[11] Felsenthal and Machover (2004b), p. 8

The following axiomatic characterization of power indices is based on Allingham (1975).

1.3.1 Basic Axioms

Axiom D (dummy member). A member $i \in N$ of the committee $[\gamma, \omega]$ is said to be **a dummy** if $\sum_{k \in S} \omega_k \geq \gamma \Rightarrow \sum_{k \in S - \{i\}} \omega_k \geq \gamma$ for any winning configuration $S \subset N$, such that $i \in S$. If *i* is a dummy, then $\pi_i(\gamma, \omega) = 0$.

Axiom A (anonymity). Let $[\gamma, \sigma\omega]$ be a permutation of a committee $[\gamma, \omega]$, then $\pi_{\sigma(i)}(\gamma, \sigma\omega) = \pi_i(\gamma, \omega)$.

Axiom S (symmetry). If *i* and *j;* $i \neq j$ are symmetric, then $\pi_i(\gamma, \omega) = \pi_j(\gamma, \omega)$.

At first glance we can see that axiom D means that a dummy player cannot benefit any voting configuration by joining it. Thus a trivial voter who is assigned zero votes is always a dummy, not vice versa.

Axiom A means that the power is the property of a committee and not of members' numbers or names; and finally S states that the power of symmetric members is the same. These characteristics are very basic and with no doubt desirable.

Axiom LM (local monotonicity). If $\omega_i > \omega_j$, then $\pi_i(\gamma, \omega) \geq \pi_j(\gamma, \omega)$. Thus the power of any committee member cannot decrease with an increase of his weight, ceteris paribus.[12]

1.3.2 Additional Axioms: GM, Additivity

No more than two power indices simultaneously satisfy all the axioms from 1.3.1. These two are based on already introduced (1.2) concepts by Penrose and Shapley and will be explicitly defined in 1.4.2 and 1.4.4. Apart from overview section 1.4 we neither consider nor test any of the other indices any further.

[12] For extensive discussion see part 2.2.2.

Here stated axioms are ways how to impose additional constraints on power indices' construction so as one obtained unique binary form of an index.

Additivity

'Let the game $G = (N, W)$ *be a composite game consisting of two constituent games* G_1 *and* G_2 *in the following sense:* $v(S) = v_1(S) + v_2(S)$*, for all* $S \subseteq N$*. Here* v *is the characteristic function of the game G and* v_1 *(*v_2*, respectively) is the characteristic function of game* G_1 (G_2)*. The additivity condition requires that the power index value of any player i in G is the sum of his power index values in* G_1 *and* G_2*.'*[13] By additivity, the only acceptable index is SSI (see part 1.4.2).

Alternatively, to obtain BCI (see subsection 1.4.4) an index should satisfy the following: '*For any two games* G_1 *and* G_2 *with characteristic functions* v_1 *and* v_2*, respectively, there is a real valued function f such that for all* $S \subseteq N$:

$$f(\max(v_1(S), v_2(S))) + f(\min(v_1(S), v_2(S))) = f(v_1(S)) + f(v_2(S)).$$

Moreover, $f(v(S)) = h(v(S))$ *for all* $S \subseteq N$*, where* $h(v(S)) = \sum_{i \in S} h_i(v)$ *= the number of swings for voter i.'*[14]

However complex, we see additivity as but another concept whose aim is to formally distinguish between Banzhaf and Penrose. Unlike, global monotonicity; though normative as well as additivity; has very strong intuitive background. It is (at least) in compliance with another implicit property of power indices we long for: to be vulnerable to the least volume of paradoxes possible.

Global monotonicity

[13] Nurmi (1987), p. 186
[14] Nurmi (1987), p. 186

Th quality required is well visible from the definition itself. *'Let* $[\gamma,\alpha]$ *and* $[\gamma,\beta]$ *be two different committees of the same size such that* $\alpha_k > \beta_k$ *for one* $k \in N$ *and* $\alpha_i \le \beta_i$ *for all* $i \ne k$*, then* $\pi_k(\gamma,\alpha) \ge \pi_k(\gamma,\beta)$*.'*[15]

Although the definition is for two different committees, we can imagine the GM condition as a situation when a member of a committee isn't worse off if he and only he is given more votes, while in the summation the same number of votes is taken away from the other members, ceteris paribus.

A well known bargaining idea is hidden behind. The only point is that marginal value of additional votes (AVV) can equal zero. It makes perfect sense till everybody is obligated to use their votes. On the other hand, if a committee member can be better off with fewer votes (GM not satisfied) it would be a relevant idea to allow their non - obligatory use. Then, of course we would need to extend our model for the possibility of abstention (what follows from part 2.3 on).

1.4. Common Power Indices

Five most widely known power indices are based on Shapley and Shubik (SSI) (1954), Penrose, Banzhaf and Coleman (BCI) (1946, 1965, 1971), Johnston (JI) (1978), Holler and Packel (HPI) (1978, 1983), and Deegan and Packel (DPI) (1978). For their interesting features we also briefly discuss Simple Proportional Power (SPP) and Majority Coalitions Ratio (MCR).

1.4.1 Simple Proportional Power

If ω_i is a number of votes of the i-th member, simple proportional power (SPP_i) equals:

$$SPP_i = \frac{\omega_i}{\sum_{i=1}^{n} \omega_i}.$$

[15] Turnovec (2003), p. 139

The power exercised by each player cannot be unfortunately[16] explained by SPP as can be seen from a simple example of: $(\gamma, \omega) = (5*10^n + 1; 5*10^n, 5*10^n -1, 1)$. Apparently, if the 1st member can form a winning coalition with whichever other player, while the 2nd cannot win without the votes from the 1st, the ratio of their power (1st to 2nd member) $\frac{5*10^n}{5*10^n - 1} \rightarrow 1$ for $n \rightarrow \infty$ cannot be accepted, whatever the concept of winning voting configurations.

Idea of SPP's Insufficiency Proof

For a priory power index a dummy player *i* with a positive number of votes ω_i in a committee; γ, ω finite positive numbers, is supposed to have $SPP_i > 0$ what contradicts axiom D according to which $SPP_i = 0$ for such a dummy.

The point is how much power the members of the given coalition have got if we turn the concept of SPP down. And that is why we need other, more sophisticated concepts of power indices.

1.4.2 The Shapley - Shubik Index

Shapley - Shubik (SSI) is defined as a ratio of number of variations[17] V_i where p_i is a pivotal player to the number of all variations V, i. e. $SSI(p_i) = \frac{V_{pi}}{V}$. Apparently, if there are n voters, V = n!. The implicit normalization of SSI stems from the fact that each variation is characterized by exactly one pivotal player. For every single voter $0 \leq SSI(p_i) \leq 1$.

[16] It would be a practical rule to allocate votes linearly to some characteristic of players, i. e. population the players represent. However, power wouldn't be proportional to those characteristics then. But still, SPP's main advantage is in general believe of its fairness and therefore wide political acceptability.

[17] SSI counts variations *k* from *n*, where *k* is vulnerable and determined by the position of pivot, as the order of elements out of any considered coalition is unimportant. For simplicity we can take all *n* elements as ordered and count permutations of *n* from *n* with the same result.

A variation is an ordered list of all members. It is assumed that all winning configurations are possible and all variations, or in other words, all orders of coalition formation are equally likely.

Even the authors of the model consider it of an illustrative one. It is assumed that the players with stronger preferences to form a winning coalition S enter the coalition formation process first and then the others join them in the order of their preferences.

Because each variance is taken into account with equal probability, this gives unequal weights to coalitions with different number of members (elements). The more elements of a given coalition, the most often its elements are considered members of a winning coalition. That makes the power of coalitions with more members overestimated.

At this point we would be tempted to turn the SSI down as the overvaluation of more membered coalitions leads to SSI being biased in its outcomes (for the same result see e. g. Holler (2002) or Loužek (2003)). Anyway, there is no other index fulfilling D, A, S, LM and GM axioms. This feature has given SSI a prominent position it occupies till nowadays.

In stead of accepting SSI, we'll hereinafter discuss the limited importance of GM in models with abstention that are according to our opinion better models of the real world than conventional power indices.

1.4.3 The Majority Coalitions Ratio

Majority Coalitions Ratio (MCR) is no more than a simple ratio of majority coalitions (coalitions meeting the quota requirement γ) containing member p_i to all theoretically possible majority coalitions.

The overall number of k combinations from n elements without repetition is $C^A = \sum_{i=1}^{n} \binom{n}{k} = \sum_{k=1}^{n} \frac{n!}{(n-k)!*k!}$, where $k \leq n$. Majority coalitions (for simple majority) are those satisfying $\sum_{i=1}^{k} \omega_i > \frac{\sum_{i-1}^{n} \omega_i}{2}$.

MCR overestimates the number of possible majority coalitions. Because all coalitions satisfying $\sum_{i=1}^{k} \omega_i > \frac{\sum_{i-1}^{n} \omega_i}{2}$ are taken into account, it is clear that in a part of these (let's denote it *f*) a notable proportion of members are so called surplus players (free riders), including dummies. If a group of coalitions *f* is less likely to form, MCR is not a good index. As we'll see in the part 1.4.5, Holler-Packel index, the answer to this question is not as easy as it seems.

1.4.4 The Banzhaf - Coleman Index

Banzhaf - Coleman (BCI) assigns each member a share of power proportional to the number of critical winning configurations for which the member is marginal. Term 'swing' is used for voters who, by changing their 'yes' to 'no', can change the coalition from winning to losing. It is assumed that all critical winning configurations are possible and equally likely.

The Banzhaf power index was originally defined by a ratio of swings of player *i* to 2^{n-1} that represents the theoretically maximal number of swings he can have.[18] The components of such an index do not necessarily sum up to unity and therefore do not provide information on relative power.

The original Banzhaf-Coleman power index (Penrose's φ_a) is often normalized and takes a form of a ratio of the number of the i-th player's swings to the total number of swings ($\sum \varphi$) of all the players.

Note that the summation $\sum \varphi$ can vary even under different decision rules of the same committee. Therefore: '*the Banzhaf index*[19] *... can only be used for comparing the voting powers of several voters under the same decision rule.*[20]

[18] Let us call the original Banzhaf Power Index non-normalized, while the one we usually use and refer to hereinafter, if not said otherwise, normalized.
[19] Normalized, note of the author.
[20] Felsenthal and Machover (2004b), pp. 4-5

For comparing the powers of the same voter under different decision rules we have to use φ_a, i. e. the original, non - normalized form of the index.

Intuitive justification of the Banzhaf index is based on a model where each player randomly votes ‘yes’ or ‘no’ with equal probability.

BCI counts with unordered combinations of elements without repetitions. That's its great advantage compared to SSI.

1.4.5 The Holler-Packel Index

Holler and Packel measure power as a proportion to the number of minimal critical winning configurations[21] (MWC) (coalitions in which all members are swing members) a player can be a member of. It is assumed that all winning configurations are possible but only minimal critical winning configurations are formed to exclude free - riding of the members that cannot influence the bargaining process. It may happen by accident that a non-minimal winning coalition forms, but it will not be stable, because in a longer time horizon all "redundant" (i. e., non-decisive) players will be removed from it. As a result, each member of MWC has the potential to turn the winning coalition into a losing one.

The idea behind HPI is clear. Why should one negotiate with potential members of a coalition if they're redundant? That is why **Public Good** interpretation is used to justify HPI.

It can be seen and Widgrén (2001) demonstrates that the Public Good Index can be written as a linear function of the Normalized Banzhaf index.

Although the public good interpretation seems luring, we present an example from political sciences that contradicts its general applicability. Let's have a three - party (A, B, C) body with allocation of votes stemming from democratic elections in the ratio of 70:15:15 with γ=71. None is a dummy and minimal critical winning configurations are (A, B), (A, C). Not only, that the last winning coalition of (A, B, C) is possible but it is

[21] Minimal critical winning configurations are also called Minimal decisive coalitions, Decisive set, Strict minimum winning coalitions. If any subset T of the winning coalition S is losing, then S is MWC. It means that S does not contain any surplus players.

quite a probable one.

The strongest party on the political market is very often in the middle, surrounded by other parties. B and C's programs are likely to be far from each other and their benefits from cooperation minimal. A can form a majority coalition with each of them. If one (e. g. B) lost, C could substitute it. If a great coalition (A, B, C) is formed, each of B, C does its best knowing its own redundancy. So, (A, B, C) is often the best alternative for A. Because both, B and C in, they are likely to diverge from their programs towards the one of A.[22]

1.4.6 The Deegan and Packel Index

Deegan and Packel (1978) measure power as a normalized weighted average of the number of MWC the player is a member of, using as weights the reciprocals of the size of each MWC. So each member *i* of MWC S receives an equal share of the coalition's power.

The aspect of indispensability seems to me general enough to be taken into account. As we can see, it is not the case of DPI, as all the members of a certain S are assumed to exercise the same power despite their basic characteristics of size. On fundamentals of LM assumption we can successfully expect bigger members of any coalition to have at least as much power as theirs smaller counterparts.

A brilliant example of LM violation[23] by DPI can be seen from the following example. For $(\gamma, \omega) = (51;\ 35, 20, 15, 15, 15)$ we get $\pi\ (\gamma, \omega) =$ (18/60, 9/60, 11/60, 11/60, 11/60). At first glance, player 2 has a smaller power value than players 3, 4 and 5, although his voting weight is higher.

1.4.7 Johnston's Index

The Johnston index (JI) takes the normalized weighted average of the number of cases when a player is marginal with respect to a critical winning configuration, using as weights the reciprocals of the number of marginalities in each critical winning

[22] For extensive argumentation see e. g. Sartori (2001), Novák (1997)
[23] DPI is not the only index that violates LM. In fact the only indices not vulnerable to the Paradox of weighted voting are SSI and BCI. We demonstrate LM violation by DPI because it is a very intuitive one. For counterexamples showing HPI and JI indices' not satisfying LM axiom see Holler (2002).

configuration. In other words, JI as well as BCI considers all coalitions in which some player is decisive. As well as the Deegan - Packel index, to each of them the weight one is assigned and divided equally among all the players decisive in that coalition.

1.4.8 Overview of Power Indices

For an overview of discussed measures of a priory power see table 1.1.

Table 1.1 Characteristics of power indices

Index	**Form of the Coefficients** $c_v(j, S)$	**Axioms Satisfied**
Shapley - Shubic	$\frac{(s-1)!(n-s)!}{n!} v'_j(S)$	D, A, S, LM, GM
Banzhaf - Coleman	$\frac{v'_j(S)}{2^{n-1}}$	D, A, S, LM
Holler	$\min_{k\in S} v'_k(S) = v(S) - \max_{T\subset S, T\neq S} v(T)$	D, A, S
Deegan - Packel	$\frac{\min_{k\in S} v'_k(S)}{s}$	not LM
Johnston	$\frac{v'_j(S)}{\sum_{k\in S} v'_k(S)}$	not LM

Sources: Malawski (2003), p. 174, Turnovec (2003), pp. 154-7, Turnovec (1998)

1.5 Extensions of power indices

Until now we have assumed equal probability of all possible voting configurations. Models under this assumption are called unconstrained.

However, it may well be the case, that not all the theoretical voting configurations' probability of occurrence is the same. If we think that there is some additional information about the distribution of voting configurations, we can introduce constrained power indices by implementing additional behavioural assumptions about voting configuration formation, cooperation, or the position in a dimensional, i. e. ideological space. As a result, not all logically possible coalitions are considered but only those we assume to be likely to occur or not all the coalitions are assumed with the same probability of occurrence.

While doing this one should still keep in mind if these (additional) assumptions are generally applicable - in other words a priory - or dependent on different sets of configurations. According to, we deal either with extensions of a priory power indices or with situation - specific, i. e. to some extend ad hoc, constrained power indices.

We're going to focus on three main ways how to extend conventional power indices:

1) Probabilistic approach to the distribution of power;

2) Spatial models;

3) Models with abstentions.

1.5.1 Probabilistic Models

If there are *n* voters and each of them supports a proposal with probability p_i; where *i* stands for a voter; we can, according to Straffin (1998), distinguish between two states or assumptions:

1) The independence assumption: each p_i is drawn from the uniform distribution $[0, 1]$ independently of the others. It means that any value between 0 and 1 is equally likely. Such an assumption suits to non - normalized BCI.

2) The homogeneity assumption: a value *p* is drawn from the uniform distribution $[0, 1]$ and p_i is set equal to this *p*, for all voters. So the probability value in the form of a real number that characterizes the entire committee is chosen randomly from the interval $\langle 0, 1 \rangle$ so that any value has the same probability of being chosen. Under the assumption of homogeneity SSI works.

In practice neither the independence nor the homogeneity assumption holds. The independence assumption omits any possible dependence of committee members, while the assumption of homogeneity goes to the other extreme. Overall, each of them gives a biased picture of 'real' distribution of power.

1.5.2 Spatial Models

Conventional power measures do not take account of institutional structures, affinities and preference structures of players, whether a voting game is of the sequential or simultaneous type, etc.

'Strategic power index is a unified method for the study of both a priori and a posteriori power; it allows the evaluation of the distribution of power within and between decision-making bodies and, most important, its sphere of application is not restricted to voting, but it can be applied to all kinds of social interactions. Furthermore, the strategic power index allows the separation of power from luck.'[24]

In the strategic (or the spatial) power index approach, the power is based on the average expected distance between the equilibrium outcome in policy games (in which players have different abilities to affect the final outcome of the decision - making procedure) and the players' ideal points for all possible configurations. The assumptions that policies can be represented by a one - dimensional outcome space, players have Euclidean preferences and information is perfect and complete are always required.

The computation of the strategic power index is only possible on the basis of an equilibrium outcome identified for each specific game and therefore it is rarely solvable.

In the case of simple voting games lacking a sequential structure, a strategic power can be reformulated as a modified Banzhaf power. It remains to be shown that such a reformulation is also possible for games with a more general structure.

[24] Schmidtchen, Steunenberg 2002, pp. 205-6

2. Paradoxes of Power?

Situations that contradict intuitive expectations about the properties of power (as we defined them) are called paradoxes. There's a whole set of these paradoxes described in this part of our diploma work, including a few newly introduced ones. We also focus on sources of paradoxical situations, i. e. whether a paradox is possible though undesired property of power as such or if it is an amazing property of certain power index's definition.

The part is called the Paradoxes of power. However, there are many more groups of paradoxes included. That's due to the fact that we'll need some of them for better understanding of the hereinafter introduced concepts. A summary of up-to-date known paradoxical situations with strong emphasize on the ones necessary for our purposes can be found is section 2.1.

2.1 Classification of Voting Paradoxes

Let us quote Nurmi, the guru of voting theory about what voting paradoxes are: '*The outcomes are bizarre, unfair or otherwise implausible, given the expressed opinions of voters. ... By their very nature some paradoxes are unavoidable. Thus, minimizing their probability is the best one can do in some cases.'* [25]

Voting paradoxes are many and their classification may vary. Apart from those we have developed and are going to introduce in part 2.3 we distinguish as many as seven basic groups. In section 2.1.1 we present for our further analyses crucial the Paradoxical act of voting, 2.1.2 deals with the paradoxes of 'stone age': Condorcet's and Borda's. 2.1.3 presents Representation and Compound majority paradoxes, while 2.1.4 comes with Monotonicity and Intra profile paradoxes. The whole part 2.2 is dedicated to Binary paradoxes of power.

2.1.1 The Paradoxical Act of Voting

[25] Nurmi (1999), p. vii

Apart from the somewhat softer No-show paradox that deals with such a situation when a part of the electorate may be better off by not voting than by voting according to their preferences, this is a one-member group of paradoxes. The paradoxical act of voting tries to answer a crucial question why so many - usually most - eligible people vote even though the effect of their vote is very marginal and they incur real costs from voting.

The Paradoxical Act of Voting, Model by Riker&Ordeshoook

Based on Downs's idea, Riker and Ordeshook (1968) came to R= PB-C+D, where R is the reward derived from the act of voting, P the probability that i´s vote will bring about the desired outcome for *i*, B the benefit *i* experiences when his favourite *x* wins, C the costs of voting and D satisfaction stemming from a variety of sources such as compliance with the ethics of voting, affirming allegiance to the political system and affirming one´s efficacy in the political system. All over, R is almost inevitably negative, unless factor D; introduced by Riker and Ordeshook; outweights R-PB+C.

Downs reasoning is that what is best depends on what the others do and ‘expect from the others' expectations' because if all the voters were better off not voting, then if I am aware of it, I will go and vote to decide. However, every other voter can think this way and therefore the model doesn't sound convincing.

The Paradoxical Act of Voting, Model by Ferejohn&Fiorina

Ferejohn and Fiorina's (1974) model itself is brilliant and the fact abstention is considered is rare in the field of voting paradoxes. The summary of the model follows.

Let C_i denote candidates, S_i states of nature and *c* costs of voting. For voter X's favourite candidate C_1 there can be distinguished five states of nature:

S_1: C_1 wins by more than one vote

S_2: C_1 wins by exactly one vote

S_3: C_1 and C_2 tie

S_4: C_1 loses by exactly one vote

S_5: C_1 loses by more than one vote

X either votes for his candidate (denoted V) or abstains (denoted A) and according to gets a pay off.[26]

	S_1	**S_2**	**S_3**	**S_4**	**S_5**
V	1-c	1-c	1-c	1/2-c	-c
A	1	1	1/2	0	0

Subjective probability distribution of S_i denoted p_i enables us to check whether the following inequality holds. If yes, it means that player X should vote:

$$[p_1(1-c)+p_2(1-c)+p_3(1-c)+p_4(1/2-c)+p_5(-c)] \rangle [p_1(1)+p_2(1)+p_3(1/2)+p_4(0)+p_5(0)]$$

After two straightforward editions one obtains:

$$p_3+p_4 \rangle 2*c$$

To quote the authors: '*Clearly, if the cost of voting exceeds 1/2, it is not rational to vote, regardless of the probabilities i assigns to states S_3 and S_4.*'.[27] Note that although the costs are in currency units this result is right because of their normalization and the fact that $1=\sum_{i=1}^{5} p_i \geq \sum_{i=3}^{4} p_i$. Thus, for X to vote, 2*c mustn't exceed $\sum_{i=3}^{4} p_i$.

2.1.2 Condorcet and Borda Paradoxes

Marquis de Condorcet (1743-1794) and Jean Charles de Borda (1733-1799) were the pioneers of the discipline.

The Borda Paradox

If the alternative ranked first by more voters than any other alternative would be defeated by every other alternative in pairwise contests by the majority of votes, we speak of Borda's paradox.

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[26] Normalized to one. Note of the author.
[27] Ferejohn and Fiorina (1974) in Nurmi (1999), p. 47

The paradox pertains to the fact that a positional procedure, the plurality voting, is not compatible with a requirement that insists on excluding the eventual Condorcet losers[28]. Borda´s way out of the paradox is to resort to a positional method named after him, the Borda count.

Simulation by Nurmi and Uusi-Heikkilä (1985) indicates (as well as for Condorcet's paradox) that the paradox's probability of occurrence is vulnerable to dramatic change with culture assumptions.[29]

The Condorcet Paradox

Intransitivity of the preference relation, formed by aggregating individual transitive preferences while using majority rule, is called Condorcet's paradox.

'The solutions' of Condorcet's paradox are always based on some form of individual opinion manipulation that is discriminating or biased. Viz. Condorcet´s successive reversal procedure, or Kemeny's[30]. Interesting due to its usage in Geneva though mistaken is Condocet's practical method.[31]

Quantification of the Condorcet's paradox probability is difficult due to the fact that entire information about voters' pairwise preferences is usually not available. The probability of Condocet's paradox can be approximated as the probability of preference profiles with no Condorcet winner.[32] For k! preference classes derived form *k* alternatives and *p* a distribution of *n* voters over these classes, a Condorcet winner exists with probability denoted $\mathrm{P}^{k}(n,p)$. Gehrlein (1997) estimates the probability of Condorcet's paradox as adjunct part to unity. He shows that the Condorcet's paradox's probability grows with the number of voters and even more with the number of alternatives.[33]

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[28] Pairwise comparisons losers.
[29] Quantification, however, strongly depends on assumptions about profiles of electorates (distribution of k! classes, called culture). Impartial culture (IC) is by far the most common assumption.
[30] Kemeny (1959), pp. 571-591
[31] See Nanson (1882), pp. 197-240
[32] The results are exaggerated to some extent since the abstention of a Condorcet winner per se doesn't imply the Condorcet paradox due to the existence of situations possessing no Condorcet winner.
[33] See footnote no. 29.

2.1.3 Paradoxes of Representation and Compound Majority

Proportional voting systems work according to a two – step scheme. First, seats are allocated to the constituencies. Second, within the constituencies, seats are allocated to the candidates. Three fundamental criteria for what to do with seat fractions should be implemented for each pair of parties:

1) **Hare minimum -** the party with more votes mustn't obtain fewer seats than rounding down;

2) **Hare maximum** - the second party mustn't obtain more seats than rounding up;

3) **Monotonicity** - the first party mustn't have more votes and fewer seats than the second.

Many methods have been introduced including Pure Proportionality (Hamilton's method) that first allocates seats according to the integers and the remaining seats to the parties with the greatest remainders. Other fundamental methods are Greatest Divisors (Hondt's method, or Minimal Distance, also known as Euclidean distance).

When Hamilton's method is used, monotonicity isn't satisfied. Such situations are described by the Alabama paradox and the Paradox of new states. The call for monotonicity is so strong that Hamilton's method isn't used any more.

Ellsberg's paradox and, as a result of experimental conclusion, in a way probabilistic Allais’s paradox form a group of paradoxes linked with utility functions.

Compound majority (CM) paradoxes pertain to the agenda of voting and are mostly caused by the presentation of issues voted upon to the voters. They include Ostrogorski´s paradox (1903), Anscombe´s paradox (1976), the Paradox of multiple elections (Brams, Kilgour and Zwicker (1998)) and the Referendum paradox. Others, based on imprecise interpretation of statistical data, are Simpson´s paradox (Cohen and Nagel (1934), Simpson (1951)), the Paradox of divided government linked with a consultative referendum and finally Schwartz's paradox of representation (1995).

A sufficient condition for avoiding CM paradoxes is that the aggregates consist of identical individuals.[34] Another way is to introduce a systematic bias such as the status quo is. It is enforced by using qualified majority rules (q-rules) that, according to the value of *q*, may also help to avoid the negative outcomes of cyclic majorities in spatial models of voting (Banks, 1995; McKelvey and Schofield, 1986; Saari, 1997).

It can be shown that by imposing $q \geq \frac{3}{4}$, one can avoid Anscombe's paradox (Wagner, 1983). Wagner also points out that there is a more general principle: '*...if the number of 'yes' entries in an n x k –matrix is at least* $(1-\alpha\beta)nk$, *then no more than* βn *voters are in the minority on more than* αk *adopted issues'*.[35]

2.1.4 Monotonicity and Intra Profile Paradoxes

Monotonicity paradoxes pertain to certain voting procedures. The least vulnerable to them are a plurality run off followed by the Borda count despite the fact that these aren't immune to the Additional support paradox. Nor are practically unused and less known procedures of Coombs, Nanson, or Dodgson (Lewis Carrol, 1832-1898).

Monotonic in this sense is the amendment procedure used e.g. in the Swedish and Finnish parliaments, or the United States' Congress. Of course, the order in which the alternatives are placed in the voting agenda is an important determinant of the voting outcome, however, supposed the agenda is kept fixed, the amendment procedure is monotonic (Fishburn, 1982). Anyway, it is still vulnerable to Pareto violations.

As well as the Additional support paradox, nor can the Preference truncation paradox be misused because it is identifiable only ex post.

While the monotonicity paradoxes are inter-profile ones, the paradoxes encountered when holding the profile constant are intra-profile ones. They include Pareto violations, the Inconsistency paradox and Choice set variance paradoxes.

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[34] However, even if the populations are not homogeneous, inferences may by accident partially eliminate or even neutralize CM paradoxes, especially the Referendum paradox.

[35] Wagner (1984) in: Nurmi (1999), p. 84

2.2 Binary Paradoxes of Power

As we know from the first part, conventional power indices are based on bipartition. We label the paradoxes that can occur under this setting 'binary' to distinguish them from the newly introduced paradoxes of abstention based on tripartition.[36]

2.2.1 The Paradox of Redistribution

One of the clearly unintuitive though quite common situations can be defined the following way:

If the power of the *i*-th member of a given committee decreases once we have increased his weight, not ceteris paribus, we encounter Paradox of redistribution.

Example 2.1. The Paradox of Redistribution

To demonstrate the Redistribution paradox, let's have a committee of (55; 50, 30, 20) compared to redistributed (55; 51, 48, 1). Player i = 1 loses power despite his voting weight increase from 50 to 51.

Numerically, for SSI the power of the first player decreases from 0.66 to 0.5 and for BCI from 0.6 to 0.5. It is well demonstrated in table 2.1. It includes all possible winning

Table 2.1. Power Values for Example 2.1

SSI		BCI	
(55; 50, 30, 20)	(55; 51, 48, 1)	(55; 50, 30, 20)	(55; 51, 48, 1)
50 30* 20	51 48* 1	50	51
50 20* 30	51 1 48*	30	48
30 50* 20	48 51* 1	20	1
30 20 50*	48 1 51*	50*, 30*	51*, 48*
20 50* 30	1 51 48*	50*, 20*	51, 1
20 30 50*	1 48 51*	30, 20	48, 1
		50*, 30, 20	51*, 48*, 1
SSI_1=4/6≈0.66	**SSI_1=3/6=0.5**	**BCI_1=3/5=0.6**	**BCI_1=2/4=0.5**

[36] See part 2.3.

coalitions for both, SSI and BCI indices.

Conditions for Occurrence of the Redistribution Paradox

It holds that for any committee (γ, ω) with *n* members; SSI or BCI; any allocation of votes *x*:

(i) If n > 3; $\gamma = \text{int}\left(\frac{t}{2}\right) + 1$ there exists an allocation *y* such that for at least one member of the committee the weight in *x* is less than in *y* and the values of both SSI and BCI are greater in *x* than in *y*.

(ii) If n > 6; whatever value of γ; there exists an allocation *y* such that for at least one member of the committee the weight in *x* is less than in *y* and the values of both SSI and BCI are greater in *x* than in *y*.[37]

Implications Stemming from the Paradox of Redistribution

The power of the member *i* of a given committee cannot be due to the existence of the Redistribution paradox supposed independent of the powers of other members of the committee. And from this fact stems the paradox. Once we know the source, i. e. oversimplified assumptions of conventional power indices that explicitly do not consider interdependence of committee members, or in other words, do not accept that power is a social concept, the situation needn't be called paradoxical.

While we have to admit the need of pronouncing the possibility of a committee members' interdependence, we are convinced, the fact that SSI and BCI compute powers according to the Paradox of redistribution is but the proof that these indices have interdependence, i. e. with no doubt a feature of the real world, implicitly incorporated. As a result we say no alternative to SSI and BCI is needed because of the Paradox of redistribution. For comparison, probabilistic approach to power indices (see part 1.6 hereinbefore) shows that interdependence of committee members contradicts non-

[37] For proof see Fischer and Schotter (1978)

-normalized BCI.

2.2.2 The Paradox of Weighted Voting

The concept of LM[38] says that the players with larger weights should never have lower power values than players with smaller weights. If LM didn't hold the cases when players with lower weights possess more power wouldn't be excluded. Such an idea is not beyond the pale for at least two reasons:

1) We've seen that there are well-defined indices that make perfect sense and do not fulfill LM, while measuring decision power as defined by these indices.

2) The concept of LM is by no means positive. We may not like indices whose outcome doesn't satisfy LM but it is not granted that those that work according to LM are better in the sense that they are better models of the real world.

Though we are not sure whether in reality decision power is always at least not decreasing with the number of votes of a committee member we assume this characteristic. Because of the general belief that LM holds players bargain for the number of votes, wish to have the highest proportion possible; and the more importance to the bargaining power we give, the most we call for LM. That is why: '*Monotonicity seems to be the watershed property to separate the adequate from the inadequate measures.*'[39]

Non-monotonicity is known as the Paradox of weighted voting. Coping with it is straightforward: use one of the indices that satisfy LM. If bargaining doesn't matter, there's no need to get rid of non-monotonicity. However, we do not know any voting body of that kind. Therefore LM is always highly desirable. It's also the reason why, from the overview of power indices in part 1.4, we stick only to SSI and BCI, i. e. to those indices that satisfy LM.

2.2.3 Paradoxes of Threshold and of New Members

[38] Local monotonicity. See part 1.2.1 for formal definition.
[39] Holler and Owen (2003), p. 36

A certain percentage limit that eliminates small parties gaining less votes than the limit is called threshold. The Paradox of threshold says that not only the excluded parties are discriminated. Even the parties that stay to be the members can be worse off.

If an addition of one new member to the committee and corresponding increase of the majority quota can improve the standing of any of the members that belonged to the committee before the alternation, we speak of the Paradox of new members.[40]

What's so paradoxical about it? In spite of the decrease of an affected member's weight he gains more power. It is well visible from the following example.

Example 2.2 The Paradox of New Members

In 1981 EU's enlargement, Greece became the only new member. It obtained 5 votes in the Council of Ministers, no re-allocation of previous members' votes took place and quota was raised. As a result, Luxembourg's non - normalized BCI rose dramatically from 0.0195 to 0.0508.

2.2.4 Paradoxes of Cooperation

Based on the same idea as constrained power indices we can consider different assumptions about coalitions formation. Thus, anti - intuitively, if some members are non - cooperative it can get all of these members better. Such a situation is called the Paradox of quarrelling members.

Having a lot in common with moral hazard and later discussed blocking power, the agenda of the Council about budget deficit of member states can be seen as highly vulnerable to a paradox of kind.

The sinner with an excessive deficit is first exhorted by the Council to restore the situation. If such steps do not take place the members of the European Monetary Union vote upon sanctions, excluding the votes of the country in question.

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[40] Compare to the Paradox of new states for often coincidence despite the difference in sources of the paradoxes.

The quorum of two thirds enables the sinners, if they are many, to block the decision by simple cooperation based on not constituting the quorum when voted about other sinners. Let's name it the Paradox of powerful sinners.

This special type of cooperation paradoxes can be easily solved by exclusion of all the countries with a budget deficit from voting about sanctions.

2.3 Abstention Paradoxes of Power

We already mentioned desirability of the possibility to abstain in section 1.3.2 when introducing the additional concept of global monotonicity into power indices. Anyway, almost the whole of the literature on voting power confines itself to binary decision rules[41] where, naturally, abstention can't be recognized.

This section is strongly linked to the third part of the diploma work where we introduce a brand new abstention model of a power index. Before that, however, we need to consider how binding the axioms from sections 1.3.1 and 1.3.2 are for its construction. At the same time, it helps us axiomize the model we have been developing. Thus, in this section we begin with axioms leading to a so far unknown abstention model (the model's shaping) what will help us later on not to contradict these axioms when the model has been introduced.

2.3.1 Back to General Monotonicity

Let's use the committees presented in Turnovec (2003) for argumentation on GM:

$$(\gamma,\alpha)=(\left(\frac{9}{13}\right);\left(\frac{6}{13}\right);\left(\frac{4}{13}\right);\left(\frac{1}{13}\right);\left(\frac{1}{13}\right);\left(\frac{1}{13}\right));$$

$$(\gamma,\beta)=(\left(\frac{9}{13}\right);\left(\frac{5}{13}\right);\left(\frac{5}{13}\right);\left(\frac{1}{13}\right);\left(\frac{1}{13}\right);\left(\frac{1}{13}\right)).$$

For better orientation let's name the members of the committees A to E and

[41] For an introduction to non-binary decision rules see Fishburn (1973), Bolger (1986), Freixas and Zwicker (2003), or any paper by Felsenthal and Machover from the list of literature.

transform the proportional values into the numbers of votes.

Having multiplied by 13 we obtain, subject to power analyses, equivalent committee sets of $(\gamma,\alpha)' = (9;\ 6,\ 4,\ 1,\ 1,\ 1)$ and $(\gamma,\beta)' = (9;\ 5,\ 5,\ 1,\ 1,\ 1)$, i. e. two committees of the same size $\gamma = 13$ and the same quota q = 9.

The example was originally used to show that BCI doesn't satisfy general monotonicity. Due to the fact that the proof is of significant importance for our purpose, we reconstruct it briefly:

The Proof of BCI's not General Monotonicity

If we count BCI for committees (γ,α) and (γ,β), as well as for committees $(\gamma,\alpha)'$ and $(\gamma,\beta)'$, we get $\pi(\gamma,\alpha) \approx (0.4737, 0.3684, 0.0526, 0.0526, 0.0526)$, while $\pi(\gamma,\beta) = (0.5000, 0.5000, 0.0000, 0.0000, 0.0000)$.

The size of the committees (γ,α) and (γ,β) is the same; for member A it holds: $\alpha_A > \beta_A$, while for $i \in \{B, C, D, E\}$: $\alpha_i \leq \beta_i$. Using BCI, we can see the condition $\pi_A(\gamma,\alpha) \geq \pi_A(\gamma,\beta)$ is not satisfied, as stems directly from $\pi_A(\gamma,\alpha) = \frac{9}{19} \approx 0.4737 < 0.5000 = \frac{1}{2} = \pi_A(\gamma,\beta)$. Therefore BCI doesn't generally satisfy GM axiom. QED

2.3.2 The Paradox of Abstention (GM under Abstention)

Now, referring to the last paragraph of 1.3.2 let's consider the abstention outcome on GM. Thus member A could, as well as the others, opt for not using all his votes. What if he didn't vote - in the example above - with the additional $6 - 5 = 1$ vote he gets from player B? His power would increase to $\frac{8}{13} \approx 0.6154$ as for BCI and $(\gamma,\alpha^*) = (9;\ 5,\ 4,\ 1,\ 1,\ 1)$ we easily calculate $\pi(\gamma,\alpha^*) = (0.6154, 0.3846, 0.0000, 0.0000, 0.0000)$.

We can see that a player (in our example A) can be better off not using all his votes!

Definition 2.1 The Paradox of Abstention

Let's call a situation when a committee member's power increases when abstaining from voting with a certain part of his votes the Paradox of abstention.

Formally: let's have a committee member *i* with corresponding $\pi_i(\gamma,\omega)$. If, ceteris paribus, $\exists\varpi_i^-$; $\omega_i^- < \omega_i$ and $\pi_i(\gamma,\omega_i^-) > \pi_i(\gamma,\omega)$ we speak of the Abstention paradox.

It is clear now that there may exist situations when abstention is rational. So a model allowing abstention would be helping extension of known power indices unless the Abstention paradox is not a characteristic of power but of a group of indices.

Whatever the answer to the previous paragraph, here it is profitable for A to abstain because of a certain votes distribution. So it is a rational source of abstention but only for constrained, i. e. not a priory power measures. A priory we consider all voting configuration with the same probability and due to the existence of situations like counterexample 2.1. (see hereinafter in part 2.3.3) we'll need other reasons to pronounce any volume of a priory abstention rational. On the other hand, having in mind the probabilistic nature of power indices, proved existence of the Abstention paradox makes us uneasy.

Ambiguous Sources of the Abstention Paradox

We are aware of the fact this section might seem odd. To let some light in, let's summarize. From the GM axiom abstention seems to be a challenging extension of power analyses. However, if abstention allowed, the Paradox of abstention could occur.

We face two difficulties. First, the Abstention paradox itself doesn't justify departing from bipartition. So, why to analyse something that, in fact, cannot occur because there are no sources of a priory abstention that would justify its allowance? We answer in the third part by presenting sources of a priory abstention.

Second, there's a need to analyse the source and type of the Abstention paradox and whether its occurrence can be eliminated by restricting ourselves to certain index, namely SSI.

Can the Abstention paradox occur under SSI as well? The answer is no. It directly stems from the axiom of GM (see section 1.3.2). If we put player A's value from committee $(\gamma,\alpha)'$ for α_k and A's value from committee (γ,α^*) for β_k, the result is that it cannot occur for indices satisfying GM. More explicitly, the Paradox of abstention cannot occur for SSI, as there is no other index satisfying general monotonicity.

Thus the nature of the Abstention paradox is in monotonicity. We are tempted to say it is index dependent as all the indices but one (SSI) are vulnerable to be non - monotonic in the sense of the Abstention paradox. On the other hand, one can also argue that BCI's not being generally monotonic and GM as a sufficient condition of no occurrence of the Abstention paradox is mere coincidence. Yet, general monotonicity is a sufficient but not necessary condition of an index not being vulnerable to the Abstention paradox.

Other Characteristics for the Abstention Paradox to Occur

Further analyses are necessary to indicate the minimum number of committee members that are needed for the paradox to occur. All we know is that this number is strictly more than two (see table 2.2.) and at most five, as stems from the suppositional committee presented in part 2.3.1.

Table 2.2. Impossibility of the Abstention Paradox's Occurrence for Committees with Two Members and BCI

Possible distribution of swings			
A B AB	A* B A*B	A B* AB*	A* B* AB

Each column represents power distribution. For the Abstention paradox to occur it should be able for player A (players are interchangable, the same applies to B) to move from at least one of these columns to the other one where he would exercise more power by not using all his votes. We can see that by this move neither one - member coalition consisting of other players that A can be negatively affected.

From the first column, there's no way out, the same applies to the third one. From the second column, there's no need to move away, while from the last one the only possible move is to column three, which is not a profitable change for A.

2.3.3 GM's Relevance for BCI under Abstention

The GM axiom is a dividing line between SSI and BCI. However, how relevant is it under abstention? Is GM as strong as for conventional binary setting, or is it a general feature of BCI under abstention that GM is satisfied?

Can the power of a member decrease when some votes are taken from another player, ceteris paribus? If not, then GM is always satisfied for BCI with abstentions. The voter that is given additional votes is then assured not to lose his power if implementing a strategy of not using additional votes (the same one that can be seen in the example from section 2.3.2). We do not say it would be the best strategy possible but it would be good enough to reassure GM.

Unfortunately, we have directly proved, by finding counterexample 2.1 that the power of a member can decrease when some votes are taken from another player. Reminding the implicit dependence of players between themselves, it was, in fact, expected.

Counterexample 2.1.

For $(\gamma,\delta) = (52;\, 25,\, 6,\, 21,\, 27,\, 21)$ and by assumption BCI we get $\pi(\gamma,\delta) = (\frac{8}{24};\frac{4}{24};\frac{4}{24};\frac{8}{24};\frac{4}{24})$. If we now take one vote away from player C to obtain $(\gamma,\delta^*) = (52;\, 25,\, 6,\, 20,\, 27,\, 21)$, we can see from $\pi(\gamma,\delta^*) = (\frac{7}{27};\frac{3}{27};\frac{3}{27};\frac{9}{27};\frac{5}{27})$ that there are other players negatively affected – player B, for example. Yet, for $\delta_B = \delta_B^*$; $\delta_C > \delta_C^*$ and $\delta_{i\neq B,C} = \delta_{i\neq B,C}^*$: $\pi_B(\gamma,\delta) > \pi_B^*(\gamma,\delta)$ because $\pi_B(\gamma,\delta) = \frac{4}{24} = \frac{1}{6} > \frac{1}{9} = \frac{3}{27} = \pi_B^*(\gamma,\delta)$.

Table 2.3. Swings for committees (γ,δ) and (γ,δ^*)

	(γ,δ)	(γ,δ^*)
	A*D*	A*D*
	A*B*C*	**ABC**
	A*BD*	A*BD*
	A*B*E*	A*B*E*
	A*CD*	A*CD*
	A*C*E*	A*C*E*
	A*D*E	A*D*E
	B*C*D*	B*C*D*
	B*D*E*	B*D*E*
	C*D*E*	C*D*E*
	ABCD	**ABCD***
	A*BCE	**A*BCE***
	BCD*E	**BCD*E**
$\sum *$	**28**	**27**

Source: Own.
Notes: changes in bold.

The initial idea doesn't hold and we cannot prove the dispensability of GM (for being always satisfied) for BCI under abstention this way. On the other hand, we have proved that BCI values depend on δ's of other members. Once again have a look at 1.2 and 1.4.4 to see how important it is, from this point of view to distinguish between BCI and its raw (or non - normalized) form.

2.3.4 Probability that GM is not Satisfied under Abstention

We haven't found a committee not satisfying GM under abstention and BCI and therefore are still unable to prove that general non-monotonicity can occur for BCI under abstention. However, what we can do is to quantify, i. e. to set the limits and probabilities of general non - monotonicity under abstention compared to situations when abstention is not possible.

It is of high importance because it will help us understand why the choice of a power index to be implemented into our model in part 3.3.3 is less important than it is for binary analyses. As is seen from ten consecutive steps that follow, GM under abstention isn't satisfied in far less situations than it is in binary models and as a result the mean difference between outcomes of SSI and BCI analyses diminishes.

Step I

Proof is undertaken for the original, e. i. non-normalized, version of BCI under abstention and is better understood if the idea of one committee with redistribution of votes (see second of part GM axiom in 1.3.2), rather then the strict definition of the GM axiom is kept in mind. Anyway, we would like to stress that we work with the original definition of GM axiom and do not alternate it at all.

Non-normalized BCI is chosen for relative simplicity of proof and for reasons from section 1.4.4.

Step II

Whatever the volume of redistribution according to GM is, it can always be segmented into n steps if a member (let's call him B) gets n extra votes. Each step represents a situation when exactly one vote is taken away from a member different from B and redistributed to B.

Step III

If member B is given one vote (let's say from member A; $A \neq B$) some players (B, A, or the others - denoted X) may become swings (we denote *) or lose their position of swing members.

Concerning A and B, there are only four possible types of coalitions:

I) coalitions containing member A only;

II) coalitions containing member B only;

III) coalitions containing both A and B;

IV) coalitions that contain neither A nor B.

Here's the list of all possible changes of stars, i. e. of swing members for players A and B and their corresponding conditions with Q denoting exact meeting of a quorum:

Ad I)

These coalitions do not contain B, so the conditions expressed in (1) and (2) hold for whatever the abstention of B is.

(1) **A gets *** (coalition without A had Q, now it has Q-1)

(2) **A loses *** (coalition had Q, now it has Q-1)

Ad II)

If B abstains, this group of coalitions is not affected.

(3) **B gets * for non-abstaining B** (coalition had Q-1, now it has Q)

(4) **B loses * for non-abstaining B** (coalition without B had Q-1, now it has Q)

Ad III)

(5) **A cannot get * for non-abstaining B** (If A didn't have *, it was because the coalition without A had at least Q and now has at least Q + 1. That contradicts obtaining * now.)

(6) **A gets * for abstaining B** (coalition without A had Q, now has Q - 1)

(7) **A loses * for non-abstaining B** (coalition without A has more votes now; coalition without A had Q-1)

(8) **A loses * for abstaining B** (coalition had Q, now it has Q-1; the size without A is non affected)

(9) **B gets * for non-abstaining B** (coalition without B had Q; now it has without B Q-1; with B it had more than Q)

(10) **B loses * for abstaining B** (coalition had Q, now it has Q-1)

(11) **B cannot lose for non-abstaining B** (coalition without B had no more that Q - 1, now it has even less; coalition had at least Q, now it has Q too)

(12) **B gets * for abstaining B** (coalition without B had Q, now it has Q-1)

Ad IV)

No change. These coalitions do not contain any player who experienced change in the number of his votes.

Step IV

There are two theoretical situations when non - GM can occur:

α) B abstains;

β) B behaves exactly as if abstention weren't possible.

Step V

Suppose that non-GM occurred under abstention (α) from step IV). Then BCI_B has to decrease by definition of GM. This condition can be rewritten as:

(I) (10) > (12)

BCI_A mustn't increase. For BCI_A's increase A would not use the vote now taken from it even before. (There is no change in the number of votes of B and Xs.)

Clearly, the only difference from the initial state of a given committee is one vote taken away from A. If it was profitable for A not to use this marginal vote it wouldn't use it in either of the situations. That is why the condition of not increasing BCI_A is necessary. The condition can be formally rewritten as:

(II) (8) + (2) $\geq$ (6) + (1)

Step VI

Suppose that non - GM occurred under β) from step IV (B doesn't abstain). Then BCI_B has to decrease by definition of GM. For BCI_B's decrease the condition is:

(III) (4) > (9) + (3)

Let's interpret (III). Condition (4) represents coalitions of type X that had Q-1 votes. Condition (9) represents situations when coalitions of type AX had Q votes.

Suppose for a while that A had initially equaled 1. Under this assumption conditions (4) and (9) would have been satisfied simultaneously. Thus condition (III) wouldn't have been satisfied with formally (4) = (9). As a result we get restricting conditions of $A \neq 1$ initially.

Similarly, for conditions (4) and (3) we get initial $B \neq 0$.

Step VII

The condition for non - gaining A under abstention (step V, condition (II)) can be rewritten as:

(IV) (10) + (12) $\geq$ (6) + (1).

(IV) can be seen directly from step III. Coalitions (10) and (8) are both of ABX type with initial Q votes; for (12) and (2) the shared parts of the coalitions were of AX type with Q votes.

Step VIII

Let's do the same steps as if we wanted to prove that non - GM cannot happen under abstention. We can try an indirect form of proof not considering situations when non - GM doesn't happen under non-abstaining B. (Whenever GM doesn't happen under non -

abstaining B, it won't happen under the possibility of abstention either because B never abstains to hurt himself and there is always the possibility not to abstain under abstention.)

Formally we want to proof:

condition (III) from step VI $\Rightarrow$ not condition (I) from step V

Rewritten:

$(4) > (9) + (3) \Rightarrow (12) \geq (10)$

From step III we can see that (12) and (9) are equivalent. Thus we reformulate to $(4) > (9) + (3) \Rightarrow (9) \geq (10)$.

Put together, we get $(4) > (9) + (3) \geq (9) \geq (10)$. This can be reformulated to $(9) \geq (10)$. (The idea behind: $(4) > (9) + (3)$ is satisfied by assumption; (3) is nonnegative by definition. Condition $(4) > (9) + (3)$ cannot contradict condition of $(9) \geq (10)$.)

If (9) ≥ (10) holds, proof is finished.

Step IX

Once again we have a look back at step III to see what (9) and (10) stand for:

(9) represents coalitions of type ABX, where the part without B, i. e. AX, had Q votes;

(10) represents coalitions of type ABX, where each of such whole coalitions had Q votes.

Step X

A priory can coalitions of type (10) form up to n - member subcommittees of the initial committee, where such a subcommittee has at least n = 3 members (at least one X; for simple reason that non GM cannot occur for committees with less than three members).

Unlike (10), AX parts of coalitions of type (9) can have only up to n – 1 members (the n-th member of the whole coalition of type (9) is B). The condition of at least one X is satisfied for the AX part having at least 2 members. Thus, at least 2 members of the subcommittee AX are required.

Summed, the difference is that subcommittees AX defined by (9) can have n − 1 members at most and can consist of as few as two members (unlike committees defined by (10) that cannot have less than three members but can have up to *n*). All other aspects are the same. The next two paragraphs analyse these two states.

The probability of any *n* - member committee to reach exactly Q votes as its summation equals zero unless Q = 100 %, i. e. unanimity.[42] For any quorum lower than 100 % and the sum of the members' votes equal unity, any *n* - member subcommittee is identical with the committee itself and has no less than 100 % of votes. Note that we speak about the group of committees satisfying (9), i. e. about the situation before abstention, where the sum of the members' votes should equal unity by assumption!

The probability of a two - member committee to reach exactly Q votes is whatever the distribution of votes decreasing with the number of committee members but non-negative by definition of probability.

Resume and Implications

Due to the backed exclusion of unanimity and for $2 < n$, the probability of (9) $\geq$ (10) equals the probability of (10) $\geq$ (9). n = 2 speaks in favour of (9) $\geq$ (10). **Thus (9) $\geq$ (10) is satisfied for any acceptable committee ($Q \neq 100$; $A \neq 1$; $B \neq 0$, at least 3 members) with the probability not lower than 0.5. QED**

We can see that though not sure if non - GM can occur under abstention; if it could, its probability would be much lower then it is without abstention and therefore the GM axiom loses part of its importance. What's more, non-monotonicity in the sense of Abstention paradox cannot be a priory misused by any member of any committee, whatever the size of the committee and whatever the number of its members, provided no other tools that power indices are used.

[42] However, there is no point in abstention under unanimity from the individual point of view. That doesn't contradict the fact from section 3.2.2 that even abstention can have non - neutral outcome.

As a result of the proof we can see that whenever non – normalized BCI is used for analyses of real – world committee data the results may be distorted. More explicitly, the results are no more that a lower bound of the voting power non – normalized BCI claims to calculate. This is due to the fact that some players' powers are underestimated, as their possibility of abstention in accordance to the Abstention paradox is neglected.

Overall, results presented here in before in part two suggest that SSI with its zero vulnerability to fail GM and the Abstention paradox is more reliable. On the other hand, the proof has showed that that non - normalized BCI is less vulnerable to non - GM than believed, which diminishes difference between the two indices.

3. Abstention

3.1 Abstention by a Binary Index?

One more paradoxical situation can be described. Let's call it the Paradox of abstention by non - abstention. The core of it is well explained by the following sentence: if abstention is rational (but not a priory) we can describe models allowing abstention by conventional power indices that do not take abstention into account[43].

The explanation stems from the very often - omitted difference between unconstrained, i. e. a priory and constrained power measures. Thus, if we introduce discrete (not a priory) sources of rational abstention, we can still use conventional power measures but only if a priory situations are analysed.

If we introduced abstention under binary setting we would face the following problems:

1) In some situations a player can seem as not satisfying GM, though it is not the case. We know from part 2.3.2 that this can be avoided by restricting ourselves to SSI, which is the only relevant - in the sense of satisfying LM - index that isn't vulnerable to the Abstention paradox.
2) Are there any sources of rational a priory abstention? If yes, does it matter that these are omitted by binary power indices? We discuss this in part 3.4.
3) If I can profit from not using some votes, what is the value of such votes for the others? Are they tradable? We do not touch this game for simplicity reasons.

Ads 1) To model the situation have a look again at player A from 2.3.1. and take (γ,α^*) as an initial committee. Then it seems that player A doesn't satisfy GM if $\pi(\gamma,\alpha^*)$ compared with $\pi(\gamma,\alpha)$. As was already shown, it is not the case. However, we do not have any tool to find the optimal number of votes one shouldn't use if maximising his power under abstention!

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[43] Compare to results from part 2.3. Yet, we can use conventional power indices but additional conditions implied by the existence of the Abstention paradox restrict us to SSI only.

We can also see that when the initial committee is (γ, β) and player B is taken 1/13 of his votes away and the same amount is allocated to player A, so as the final shape of the committee were (γ, α), player A would then opt for (γ, α^*) and the condition of GM would be satisfied. (Player A is better off from 0.5000 to $\frac{9}{13} \approx 0.6923$, while the others are not better off - B fell down from 0.5000 to $\frac{5}{13} \approx 0.3846$ and the powers of C, D and E stay unchanged.)

3.2 The Outcome of Abstention

3.2.1 Puzzling Abstentions

Let us stress once again that the existence of situations we have invented and called the Paradox of abstention does not necessarily mean that there are any rational sources of a priory abstention.

It may be the case that conventional indices work well under abstention[44] with the only exception. They fail to detect situations when posterior abstention is rational. This and only this failure, however, doesn't mean that once we have found a tool do detect situations called the Abstention paradox, conventional power indices will be useless measures of post-electoral power[45]. Till it is done, let's describe in chapter 3.2.2 how complex and flustered the outcome of abstention is in real world committees and let's have a look if there's any institutional change that might prevent the Abstention paradox from occurring.

Nowadays, what would seem easier than analysing the set of all occurrences of the Abstention paradox under a certain distribution of votes by numerical computing? However, reality contradicts this. In the bodies of shareholders, for instance, it might be

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[44] We'll further see that this hypothesis doesn't hold. It may have held, however, if there were no rational sources of a priory abstention.

[45] See previous note.

Theoretically, two approaches are possible for the incorporation of posterior abstention into our model. First, we can stick to (probably constrained in assumptions about distribution of majority coalitions) conventional indices and find out some tool or method to detect the Paradox of abstention. Second, we can introduce a brand new index that wouldn't allow paradoxes of this kind. Neither of these approaches seems to be easily solvable and further research is needed.

quite time consuming as the number of plausible distributions can easily reach up to 10^{1000}.[46] What's worse, we face a game that is simultaneous (all members vote at the same time). Thus, if I know that somebody is better off abstaining I should also try to abstain to reduce his advantage.

The core of the problem is that some abstentions can possibly be neutralised by other players, some cannot. What to do with the set of abstentions that cannot be neutralised? Do the players that have such luck posses some extra power? How rational and well equipped are holders of a few single shares for an analysis of their abstention strategy?

3.2.2 Posterior Abstention and its Outcome

When analysing abstentions in models that are not a prior, we have to keep in mind the distinction between active and passive abstention. Passive means that a member of a committee doesn't turn up at the act of voting, while active stands for situations when he's present but actively votes for 'abstain'.

Taking into account the two most commonly used quota settings that count quota as a ratio of all members of a committee, or present ones, respectively; we obtain a two - space model with four possibilities (see table 3.1).

Table 3.1 Quorum - Abstention Overview

No. of Alternative	Members Quorum Calculated from	Type of Abstention
I	All (We denominate 'fixed quorum')	Passive
II	All (We denominate 'fixed quorum')	Active
III	Present	Passive
IV	Present	Active

Source: Own.

Alternative I: Passive abstention doesn't change the quorum. Therefore, if

[46] Based on Richter (2002), who is concerned with the power of attorney difficulty, we can generalise and pronounce a mean number of plausible distributions in any body of shareholders diminishing with the maturity of the capital market.

abstaining or voting against, the result of polls is perfectly the same.

Alternative II: Active abstention doesn't change the quorum either. In spite of the fact a voter wasn't against, his vote is treated so.

Alternatives I&II:

The first alternative as well as the second have one very important feature in common. They are used; or at least generally believed to be appropriate; only for very important constitutional decision making. We strongly disagree with the way it is enforced.

Meeting a fixed quorum is required to protect status quo from too many or not well - thought - up changes. For such cases the quorum is usually higher than simple majority[47] and to be fixed is considered as a necessary condition for ensuring it. The problem we see in is fragrant discrimination of committee members who decide to abstain.

Alternative III: The member doesn't come to take part in voting. Thus the number of present committee members diminishes and as a result a lower quorum is required.

At first glance, with the lower quorum the probability of a bill to be accepted gets higher. It seams that the outcome of passive abstention isn't neutral. It is the case but from a different source that it seems to.

First, a motivation example: clearly, for normalised quorum q_i, non - normalised quorum q^n, ratio of present electors k_i and number of electors *m*; one can take a hypothetical committee with $q_1 = 0.51$, $k_1 = 1$ and m = 100, each member having one perfectly divisible vote. Assume that a bill passed with just a minimum of $q_1 * k_1 = 0.51$ support, i. e. all members were present but only 0.51 * 100 = 51 = q_1^n of them voted 'for'.

If one committee member decided to abstain, what would be the effect? The adjusted committee would have $k_2 = \frac{100-1}{100} = 0.99$ ratio of present electors with unchanged normalised quorum $q_1 = 0.51$. The real number quorum (non-normalised) is obtained from $q_1 * k_2 * m = 0.51 * 0.99 * 100 = 50.49$. As a result we can see that the

[47] It helps to reduce threat of compound majority paradoxes too; however, it's rarely touched upon by politicians who are more likely to be affected by 'blocking power'.

quorum diminished for 51 - 50.49 = 0.51 votes. That is the quorum diminished in the ratio of the normalised quorum.

General proof

Non - normalised quorum q^n is a linear function described by:

$$q^n = f(q_i, k_i, m) = q_i * k_i * m.$$

Change of q^n as a result of change in k_i is expressed as:

$$\frac{\partial q^n}{\partial k_i} = \frac{\partial (q_i * k_i * m)}{\partial k_i}.$$

Having no other assumptions we declare m and q_i exogenous and independent of k_i. As a result $\frac{\partial q^n}{\partial k_i} = q_i * m = \text{cons}.$

Reinterpretation is straightforward: passive abstention has almost neutral effect on the outcome of polls due to the fact that it lowers non - normalised quota for the share of votes equal to the value of the normalised quorum. QED

An exception from the neutral effect is encountered if the constraint of constituting a quorum is active.[48] In other words, passive abstention has a very similar but not equivalent effect as voting with the abstaining votes in the ratio of quorum 'for' and with its supplement to one 'against'.

Alternative IV: We know from alternative III that passive abstention isn't perfectly neutral in its results but near to be. On the contrary, active abstention doesn't lower down the quorum if abstaining members are computed as present ones. Yet, they are computed this way in practically all known committees including both chambers of the Czech Republic's Parliament as well as in the Commission of the European Union.

[48] See footnote no. 51.

How active abstentions affect the quorum required in decision making undertaken by the Commission in not explicitly mentioned in the Rules of Procedure of the Commission.[49] We made an enquiry and here is the answer provided:

'The provision concerning votes in the internal rules is: Commission decisions shall be adopted if a majority of the number of Members specified in the Treaty vote in favour. This majority shall be required irrespective of the tenor and nature of the decision.

This means that a Member that does not vote, does not count in reaching the number of 11 votes necessary for adoption of a decision by vote (11 being the majority in the current Commission).

As far as the quorum of the meeting is concerned, 11 Members shall be present. However, votes are rather rare in the Commission and most decisions are taken without the need for having a vote.' [50]

Such an outcome is unjustifiable and is a specimen of penalty for honest and sincere revelation of one's opinion. Such members' votes are treated as voting against, unlike those of abstaining passively, what is discrimination contradictory to the call for democratic elections.

3.2.3 Suggestions to Posterior Abstention

Fixed quorum

Once we agree with the necessity of certain status quo's protection, the fixed quorum (demanded under alternatives I and II) should be strictly higher than 50 % from all.

Alternatively, we are interested in the share of electors who are either against or do not care (passive abstention). Any elector who is physically present and actively votes for 'abstention' has exactly the same characteristics as any other physically present member actively voting for either 'yes' or 'no'. The same requirements that apply to 'yes' and 'no' voters apply to 'abstain' voters, i.e. actively abstaining voters, too. Unfortunately, we cannot

[49] Official Journal L 308, 08/12/2000 P. 0026 – 0034, on-line version from <[C(2000) 3614], http://europa.eu.int>.
[50] Kersting (2004)

Instead of the usual system of a fixed quorum we suggest a system where votes 'yes' are decisive, votes 'no' together with votes of passively abstaining are considered as voting against and votes 'abstain' lower down the quorum without having any other impact.

Passive abstentions are considered as voting against and therefore do not change the quorum so as to protect status quo. This rule has been in fact used. Votes of actively abstaining should be taken into account as a ternary and equally rational decision. It expresses one's opinion: 'whatever the result of this decision, status quo and proposed change are equal for me'. And, of course, votes of actively abstaining should be treated as those of any other present member.[51]

We are sure our suggestion will raise lively discussion and we welcome it. It is beyond our aim to dwell into each probable reservation of it here. Anyway, one apparently strong argument, is in our favour. The decisions where a fixed quorum is required are very important ones and they are undertaken after profound discussion of the problem. The members who decide (about the proposals) are well acquainted with the fact of necessity to protect status quo from low quality proposals.

If we should be afraid of somebody, it would be of a voter who is responsible enough not to vote for the proposal when he is not convinced about it (this ‘threat’ we face under current setting with a fixed quorum). And, at the same time, the voter irresponsible enough to actively vote 'abstain' when he can see a difference in favour of status quo (this is the ‘threat’ of our proposal)[52]. Thus, it would be really difficult, if not impossible, to think of such a voter. What’s more, thoughts of constrained rationality of decision - makers are irrelevant because they apply to all electors, not only to actively abstaining ones.

Present members

If alternatives III and IV are compared, one might oppose that having not to come to achieve near to a neutrality result if one's wish is to stay neutral, is unfair. All members of

[51] That’s the difference between active and passive abstention. The passive one has a negative effect on constituting the quorum and is not therefore neutral in its result. It raises the probability of a proposal not being passed (for a committee not having enough present members and therefore not constituting a quorum) compared to active abstention, which is, under our proposal, neutral in its result.

[52] Let us remind that really irresponsible voters, who passively abstain, are irrelevant to this because they are treated as voting against.

any committee should posses the same right to participate in the act of voting and being for, against or really neutral, not by merely label, but in result. We agree, noting apart from this that there are; though hidden to the formal analyses; important factors of having the same opportunity to be seen at work (to be seen voting) because it may affect committee members' probability of re - election.

The following example shows why we call for an actively abstaining committee member lowering the quota down. By this approach we would give equal rights to all no matter what their opinion and so ensure one of the very basic conditions of democratic elections.

Let's consider a committee with $q^n = 75 * k_1$; m = 100 and compare two plausible voting results in the form of (Y; A; N) standing for its members' votes for, abstention (active!) and against. The results are: $(60; 0; 20)_1$ and $(60; 20; 20)_2$. The former represents a situation when 20 out of 100 committee members decided not to vote at all (passive abstention). The quorum is $q_1^n = 75 * \frac{60 + 20}{100} = 75 * 0.8 = 60$ and a bill was approved. In the latter, contrary to the former, those members who wished to abstain decided to do it openly. Thus, $q_2^n = 75 * \frac{60 + 20 + 20}{100} = 75 * 1 = 75$ and the bill was not approved, because there were only 60 members voting in favour of the proposal.

As a result we strongly recommend allowing active abstention with the rule of abstaining members lowering quorum down. If our proposal were accepted, the difference would be between the outcome of passive abstention in I and III. It would lower the quota but for very important cases where status quo is considered superior to the simple majority's opinion. In these cases (those, when now results are calculated from all members of a committee) a decision making process would be protected from constrained rationality of electors who prefer passive abstention (that we proved not to be a perfect substitute of the active one) and from undesired results caused by the Paradoxical act of voting. Actively abstaining members would have to be counted as present ones.

The second difference is tied with the Abstention paradox as defined in part 2.3.2. If active abstentions lowered quorum down, it wouldn't occur because the paradox is tied with abstentions that do not affect quorum.

On the other hand, even though the Paradox of abstention wouldn't occur because q^n would indirectly change with the scale of active abstention, it isn't a way to avoid any monotonicity problems of this kind. Similarly to the Abstention paradox we can define the Abstention paradox II for active abstentions lowering quorum down.

Definition 3.1 Paradox of Abstention II

Let's call a situation when a committee member's power increases when abstaining from voting with a certain part of his votes under the rule of quorum lowering active abstentions the Paradox of abstention II.

Formally: let's have a committee member *i* with corresponding $\pi_i(\gamma,\omega)$ under the rule of quorum lowering active abstentions. If, ceteris paribus, $\exists \varpi_i^-$; $\omega_i^- < \omega_i$ such that $\pi_i(\gamma,\omega_i^-) > \pi_i(\gamma,\omega)$ we speak of the Abstention paradox II.

Once again we can use the committee $(\gamma,\alpha)' = (9;\ 6,\ 4,\ 1,\ 1,\ 1)$ from section 2.3.1 and its derivative by partial abstention of the first member $(\gamma,\alpha^*) = (9;\ 5,\ 4,\ 1,\ 1,\ 1)$.

We already know $\pi(\gamma,\alpha)' = (0.4737, 0.3684, 0.0526, 0.0526, 0.0526)$. For newly calculated $\pi(\gamma,\alpha^*) = (0.5000, 0.5000, 0.0000, 0.000, 0.0000)$ The Abstention paradox II is encountered due to the fact that the first player is better off.

When the probability of active a priory abstention is quantified (for opening of the discussion see section 3.3.4), we will be able to re - calculate the demanded quorum so as those processes where a fixed quorum has been used so far were a priory as vulnerable to change status quo as they are now.

To make it more understandable, with active abstentions lowering the quorum down we'll have to moderate q^n if we want on average have the same amount of status quo changes. How much we should, it depends on the strengths of rational sources of a priory abstention, we discuss in section 3.3.4.

3.3 Modelling Indices with Active Abstention

There have probably been only two teams working on abstention voting games: Felsenthal&Machover and Braham&Steffen. When we consider the fact that even the old Romans used to recognise ternary division absolvo, condemno and non liquet, we are astonished.

Due to the fact that even nowadays there are committees, including both chambers of the Czech Parliament and the Commission of the European Union, where active abstention is allowed; yet, at the same time distorted; we express our opinion that politicians rather do not know (or are content with the way it is) how to treat active absentees than that they would head towards the ban of active abstention.

If abstentions occur in real committees, wouldn't it be not only relevant but also highly desirable to model them hand in hand with votes for and against?

3.3.1 Up – to – date Existing Models

Conventional voting games are based on bipartition and always assume something similar to: *'Each committee member i's opinion can be represented as a complete and transitive binary preference relation R_i over A.'* [53]

We've shown this assumption is too simplifying for modelling real - world voting bodies where abstention is recognised. This unsuitability might be of negligible scale if votes of absentees were neutral in their outcome. However, this basic condition of democratic elections doesn't hold and provided we wish to recognise ternary division, a change is needed.

[53] Nurmi (2003), p. 48

Felsenthal&Machover propose the idea of tripartition (or ternary division) of a set A into a map T of values $\{-1; 0; 1\}$ standing for not, abstention and yes. By their straightforward approach we get for the Shapley - Shubik index of voter *i* in a tripartition (or TVG) υ:

SSI $(\upsilon) = |\{i = piv(R;\upsilon)\}| / 3^n n!$; R denotes space.

Similarly, for BCI is obtained:

'(i) The Banzhaf score or 'raw' Banzhaf index (η) of voter i in a TVG υ is the number of coalitions in which i is critical.

(ii) The Banzhaf index (ß) of voter i in a TVG υ is obtained from η_i by normalisation: $\beta_i(\upsilon) = \eta_i(\upsilon) / \sum\{\eta_i(\upsilon) : j \in N\}$, so that $\sum\{\beta_{ii}(\upsilon) : j \in N\} = 1$

(iii) The absolute[54] Banzhaf measure (β') of voter i in a TVG υ is defined by $\beta' = \eta_i(\upsilon) / 3^{n-1}$.'[55]

This rather simple model emerges from the assumption of p(Y) = p(N) = p(A), which even to the authors doesn't seem natural.

3.3.2 Extension of up – to – date Existing Models

What we add is that neither p(Y) = ½, p(N+A) = ½ that corresponds to the majority of used decision making procedures is plausible. It is due to the fact that each possible solution either contradicts p(Y) = p(A) or leads to p(N) = 0, for which we wouldn't need to alternate the indices.

Nor do we agree with the concept Braham&Steffen opened, i. e. a voter first decides to take part in a decision making process and then, if decided to do so, chooses only from yes or no. Such a concept contradicts the requirement of the non - discriminatory possibility

[54] Absolute stands for non-normalised. Note of the author.
[55] Braham and Steffen (2002), p. 338

of public abstention and is irrelevant to power measures both, under a quorum calculated from present electors[56] as well as under a changed system of fixed quorum[57] we suggest.

Table 3.2 shows the difference between the bases of these two models and the one we promote.

Table 3.2 Voter's Decision Process about the Act of Voting

Time	Felsenthal&Machover			Braham&Steffen			Hokes[58]			
I	Yes	Abstain	No	Vote		Abstain	Vote			Do not vote
II				Yes	No		Yes	Abstain	No	

Sources: Braham&Steffen (2002), own.

In fact, if it comes to the base, we are very close to Felsenthal&Machover's. They, as well as we do, recognise both, active and passive abstention[59] but then they do restrict their model to active abstention only. We say, passive abstention must be considered as well because it may be rational and very often is, viz. 2.1.1, the Paradoxical act of voting. Difficulty arises when passive abstentions do not lower the quorum down (alternatives I and II from table 3.1, the so called fixed quorum rule). In such cases it holds under the current institutional setting (once again see the Paradoxical act of voting):

$$\text{a priory } p(Y) < p(N) + p(\text{Do not vote}) \Leftrightarrow \text{post - electoral } p(Y) < p(N^*);$$

N* denotes the summation of votes against and passively abstaining ones that are, as we already know, counted as votes against.

We are convinced, this problem needs detailed in look. However, we're afraid the quantification of passive abstention, apart from the part it has in common with the Paradoxical act of voting, is more challenging for sociologists, as it really is; apart from this

56 Passively abstaining members lower the probability of constituting a quorum but do not have any other effect on the decision voted about. Their possible effect on not constituting a quorum has to be seen as accidental. It's a result of other source of status quo protection - the setting of a minimum constituting the quorum.
57 We see fixed quorum procedures as fragrant discrimination due to the fact that active abstentions are calculated as votes against.
58 Do not vote in our model corresponds with passive abstention. Here we use more instructive 'do not vote'.
59 In Machover's terminology abstention by default.

paradox; an outcome of constrained rationality of those who decide not to influence the results of polls.

3.3.3 The Power Index for Active Abstentions

Not contradicting any of the authors mentioned we can place Y, A (active abstention) and N on one-dimensional spectrum. On the line, A looks like a mere dot, Y and N are rays. It would correspond to p(A) = 0 because $\frac{1}{\infty} = 0$ and there wouldn't be any place for active a priory abstention. That's, in fact, an implicit assumption of any conventional power index, as for p(A) = 0 one can exclude A and the result of p(A) = 0 is more realistic then the result of ternary division, p(A) = 1/3.

The problem is that the human race is unable either to recognise or to distinguish between infinitely many different states on one axis. A man can distinguish far less states. While few authors of public choice books insist on sharp inequalities when it comes to individual alternatives' ordering[60], the majority does not. Sociologists, as a group of the most relevant scientists to this, strongly recommend having an odd and very limited number of alternatives in inquiries with the middle one expressing neutrality.

Anyway, while the assumption of limited abilities to compare very near and almost similar proposals is important to status quo because it defines the value of our model's parameter *p*, it isn't a fatal one.

Apart from calculating with abstentions, the power index we promote to our model is based on exactly the same methodology as usual BCI. Each member of a given committee votes with all his votes uniformly; the number of swings in winning coalitions is what decides about a member's a priory power. That's why the suggested index is axiomatically as good as BCI. It satisfies D, A, S and local monotonicity.

Because active a priory abstentions are considered, not all the winning coalitions are equally likely to occur. It is rather the way that a priory abstentions are equally likely to occur for every member of a coalition with a probability given by the parameter *p*.

[60] Let's note that it is done so because is simplifies further analyses.

Thus, we first have to calculate the distribution of abstentions as a function of externally given *p* (step I), then for each alternative analyse the corresponding committee and its quorum (step II) and calculate usual BCI for such a committee (step III). Finally, we obtain the index as a weighted summation of partial BCIs from the previous step (step IV). How might be the parameter *p* set follows in part 3.3.4.

Step I: Distribution of Abstentions

For a given and independent[61] probability *p* of each member of a committee to actively abstain and the committee consisting of *n* players we can write:

there are $\left(\frac{n!}{0! * n!}\right)$ possible settings where none of the committee members abstains and each of them occurs with the probability $(1-p)^n * p^0$.

Clearly, for independent phenomena with the same probability of occurrence *p*, $(1-p)^n * p^0$ is the probability that exactly *n* of them will not occur, while 0 will. $\left(\frac{n!}{0! * n!}\right)$ is the combination number of groups containing zero members out of *n*.

Similarly, we can write that there are $\left(\frac{n!}{1! * (n-1)!}\right)$ settings where one (certain and explicitly chosen) member abstains; each of the settings occurs with the probability $(1-p)^{n-1} * p^1$; $\left(\frac{n!}{2! * (n-2)!}\right)$ settings where two members abstain; each of them occurs with the probability $(1-p)^{n-2} * p^2$, ... till $\left(\frac{n!}{n! * (n-n)!}\right)$ settings for *n* abstaining members; each of the settings occurs with probability $(1-p)^{n-n} * p^n$.

[61] A priory implies that each player has the same information about behaviour of other players. Because a priory power indices do not count with strategic behaviour of players, this information is, in fact, none. Thus, the abstention of one is independent of the abstention of the others.

The summation of all probabilities by their corresponding combination numbers ($(1-p)^n * p^0 * \left(\frac{n!}{0! * n!}\right) + (1-p)^{n-1} * p^1 * \left(\frac{n!}{1! * (n-1)!}\right) + \ ... \ + (1-p)^{n-n} * p^n * \left(\frac{n!}{n! * (n-n)!}\right)$) equals unity. It stems from the fact that there are no other possible settings we haven't mentioned.

Step II: Corresponding Committees and Their Quorums

For the possibility with no abstaining committee member we can use common BCI.

For the possibility with one abstaining member, we get $\left(\frac{n!}{1! * (n-1)!}\right)$ = n different settings. Each of them doesn't contain exactly one committee member. For each of the settings we calculate the corresponding quorum as a required share to the votes of n – 1 members that do not abstain.[62]

Similarly, we do the same operation for all the settings, whatever the number of abstaining members. Generally, for *x* abstaining members, the corresponding quorum is the required share of votes to the votes of n – x members that do not abstain.

Step III: Calculation of BCI for Each Setting

For each setting, we do proceed as for a usual committee and usual BCI. We list all possible winning coalitions and mark the swings. The number of swings of each member is normalised with the summation of swings. The members that abstain in a particular setting are allocated no power.

Step IV: The Final Index Obtained as the Summation of Results from Step III

For each committee member, his power is obtained as a weighted summation of powers calculated in step III. Weights naturally equal to the probabilities of the settings (see step I).

[62] Note how distorting the requirement of fixed quorum would be.

Remarks

The model is based on probabilistic interpretation of power indices, what doesn't contradict the methodology of power indices. For tripartition we obtained a development equal to the binomic distribution[63] with parameters *n* and *p*, where parameter *n* is given by the number of the committee members and *p* is an exogenous value of expected active abstention. Its mean value equals n*p and variance equals n*p*(1-p).

3.3.4 How much is p?

'Life for value of parameter!'
Unknown scientist

Parameter *p* is endogenous. For empirical indices we can use various methods how to set its value. One is based on sociological scaling, what would suggest very high values, up to 1/9 and definitely not less than a few percents.

Other can be an empirical observation, which should be based on data from committees where active abstention has a neutral effect on outcome, or at least very near to. Here institutional change we proposed in 3.2.3 plays its role. So far, we have found no more than two committees very near to satisfaction of this requirement. Neither of them gives enough relevant data for an econometric model.

1) According to the German Constitution, article 63, if a counsellor isn't elected new polls take place. A candidate that obtains the highest number of votes is then elected, no matter how high the number of votes is and irrelevant of votes for the others.

2) In the Council of the European Union: *'Abstentions by Members present in person or represented shall not prevent the adoption by the Council of acts which require unanimity.'*[64]

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[63] For $n \geq 30$ and $p \leq 0.1$, binomial distribution is well approximated with Poisson's $P(x) = \frac{\lambda^x}{x!} e^{-\lambda}$ for x = 0, 1, ...

[64] The Treaty Establishing the European Community, article 205, point 3 in: Official Journal C 325, 24 December 2002, on-line version <http://europa.eu.int/eur-lex/en/treaties/dat/EC_consol.html>

For a priory indices, the situation is much more difficult. The patient reader might have noticed that for p = 0 the model we introduce is simplified to usual BCI. Similarly, we could have implemented SSI too. Thus the index proposed is a generalised version of the two basic concepts from 1.2. As such, it is vulnerable to all the paradoxes and strains discussed hereinbefore.

For the a priori version we have to quantify the value of *p*. Yet, it cannot be done without the use of empirical research and therefore distinction between a priori and posterior loses its sense.

Anyway, how relevant is information obtained by the special case of the proposed index obtained for p = 0? How relevant are all the results of SSI or BCI? It depends on the share of information omitted by these, which in turn depends on *p*, and homogeneity of this information with the information binary indices of power use. The less homogeneous, the least valuable SSI and BCI are.

For an arbitrary chosen ratio *x* of information not taken into account not exceeding 0.05 and a twenty five - member committee such as the Council of Ministers of the European Union 25 is, we get for the estimate called the binary index of power:

$$\left(\frac{n!}{0! * n!}\right) * (1-p)^n * p^0 \geq 1-x \text{; n = 25; x = 0.05} \qquad \text{(I)}$$

$$(1-p)^{25} * p^0 \geq 1-0{,}05$$

$$1\text{-}(0{,}95)^{1/25} \geq \text{p}$$

$$\text{p} \leq 2.0496 * 10^{-3}$$

We doubt whether such a value of parameter *p* corresponds to reality. Our hypothesis; which will have to be supported by empirical research; is that from a certain number of committee members *p* is too high for binary indices of power to give relevant and stable results.

3.4 Application: the Lower House of the Czech Parliament

As it's usual these days, any paper that presents a new model should include its application. We have chosen the Lower House of the Czech Parliament to be an extremely interesting body due to the results obtained.

Data used originate from the official web page of the Chamber of Deputies[65] of the Czech Republic and are up - to - date 30/04/2004. We implement them into our general model of the power index under tripartition presented in part 3.3 and compute for normalised BCI and the simple majority rule without additional requirements of meeting the quorum.

There are five political parties that currently have got their deputies in the Lower House and one unclassified member. Their names, abbreviations of names and the numbers of votes that equal the numbers of their deputies can be seen in table 3.3.

Table 3.3 Parties of the Czech Chamber of Deputies

Party Name	Abbreviation	Votes
Czech Social Democratic Party	CSSD	70
Civic Democratic Party	ODS	57
Communist Party of Bohemia and Moravia	KSCM	41
Christian Democratic Union - Czechoslovak People's Party	KDU-CSL	21
Freedom Union - Democratic Union	US-DEU	10
Unclassified	Unclassified	1

A committee consisting of 6 members with the summation of 200 votes and the quorum equal 101 votes can describe the Chamber of Deputies:

$$(\gamma,\alpha) = (101;\, 70,\, 57,\, 41,\, 21,\, 10,\, 1).$$

First, we calculated common, i. e. binary, BCI to have a benchmark with which we could compare the results:

$$\pi(\gamma,\alpha) = (\frac{5}{13},\frac{3}{13},\frac{3}{13},\frac{1}{13},\frac{1}{13},\frac{0}{13}) \approx$$

[65] http://www.psp.cz/

$\approx (0.384615; 0.230769; 0.230769; 0.076923; 0.076923; 0)$.

Second, we proceed according to the four steps from part 3.3. For the distribution of abstention we get:

$\left(\frac{6!}{0! * 6!}\right) = 1$ setting where none of the committee members abstains; its probability is $(1-p)^6 * p^0$,

$\left(\frac{6!}{1! * 5!}\right) = 6$ settings where one of the committee members abstains; each of them with the probability $(1-p)^5 * p^1$,

$\left(\frac{6!}{2! * 4!}\right) = 15$ settings where two of the committee members abstain; each of them with the probability $(1-p)^4 * p^2$,

$\left(\frac{6!}{3! * 3!}\right) = 20$ settings where three of the committee members abstain; each of them with the probability $(1-p)^3 * p^3$,

$\left(\frac{6!}{4! * 2!}\right) = 15$ settings where four of the committee members abstain; each of them with the probability $(1-p)^2 * p^4$,

$\left(\frac{6!}{5! * 1!}\right) = 6$ settings where five of the committee members abstain; each of them with the probability $(1-p)^1 * p^5$,

$\left(\frac{6!}{6! * 0!}\right) = 1$ setting where one of the committee members abstains; its probability is $(1-p)^0 * p^6$.

For each of 64 possible settings we calculate the corresponding quorum (see the enclosed microfloppy). For the possibility with no abstaining committee member we can use common BCI. The calculation of BCI for each such setting and the final index as a weighted summation of these follows.[66]

Table 3.4 The Czech Chamber of Deputies: Power Values

Value of *p*	CSSD	ODS	KSCM	KDU-CSL	US-DEU	Unclassified
0	0,384615	0,230769	0,230769	0,076923	0,076923	0,000000
0.008512	0,382448	0,232527	0,229389	0,077842	0,077794	4,43E-11
1/9	0,372980	0,251825	0,213654	0,084194	0,077330	1,51E-05
1/3	0,381985	0,281297	0,191880	0,089566	0,051157	0,002743
0.5	0,350338	0,283698	0,184161	0,101318	0,049234	0,015625

In table 3.4 we present the results for different values of parameter *p*. Value $p = 0$ corresponds to common BCI. $p = 0.008512$ is obtained for the critical value x = 0.05 from unequation (I), section 3.3.4.[67] One ninth is the value we consider being the most probable. One third corresponds to the implicit assumption of tripartition. $p = 0.5$ is an extreme case we include for a comparison.

This is an interesting result as values of the index are very stable whatever the values of *p*. However, neither is it a general property of the index[68], nor are the values monotonous in the sense of a mathematical analysis. It implies that whenever binary indices of power are used, it would be reasonable to test their stability in respect of parameter *p*.

[66] Values of parameter *p* can be changed in sheet 1, column Z. You can also calculate constrained power indices by changing values of the parameter only in some rows. Simple BCI for all the subcommittees are calculated in sheet 2.

[67] For $\left(\frac{n!}{0! * n!}\right) * (1-p)^n * p^0 \geq 1 - x$; n = 6; x = 0.05 we obtain $(1-p)^6 * p^0 \geq 1 - 0{,}05$, which leads to $1 - (0{,}95)^{1/6} \geq p \Leftrightarrow p \leq 0{,}008512$.

[68] For instance, let's have a committee $(\alpha, \beta) = (51;\ 27,\ 26,\ 24{,}23)$. While binary BCI indicates for the first member (let's denote him A) $\pi(A) = \frac{5}{12} \approx 0.4167$, with $p = \frac{1}{9}$ we get $\pi'(A) \approx 0.3678$. Similarly, for the member with 23 votes (member D) we obtain $\pi(D) = \frac{1}{12} \approx 0.0833$, $\pi'(D) \approx 0.1304$, respectively.

Conclusion

We provided a survey of power indices' methodology as is presented in recent literature and identified and analysed a new anti - intuitive property of so called 'binary' power indices: the Paradox of abstention.

The Paradox of abstention's core is in the fact that by a systematic abstention (not using part of a committee member's votes in weighted voting), providing the absolute quota remains unchanged, the a priori relative power of the member measured by Banzhaf - Coleman index might increase. It's shown that the Shapley – Shubik index isn't vulnerable neither to the Paradox of abstention nor its extension for an adjusting quota.

We have contributed to the theory of power indices by proving that the existence of the Abstention paradox lowers the importance of general monotonicity and as a result the non - normalised Banzhaf - Coleman index and the Shapley - Shubik index give more similar results.

In the thesis we show that the problem with the Paradox of abstention and traditional power indices lies in the fact that the model used in the definition of a priori voting power does not distinguish between abstention and vote against. Formally, partial abstention (not using a part of votes) creates a new committee with a different structure. To handle the abstention problem we need a new model, taking into account 'tripartition' of outcomes: YES, NO and ABSTAIN. I am proposing such a model and a new power index based on it.

It is shown that such a 'tripartition' model can be simplified to a simple 'binary' one provided the value of endogenous parameter p – representing the volume of abstention - is set zero. However, with abstention allowed, this is unlikely, which doesn't contradict the fact that for certain distributions of votes this simplified 'binary' model with the Shapley – Shubik index can by chance provide a good estimate of a priory power.

Such a committee, where the scale of abstention doesn't matter, is the Parliament of the Czech Republic, which is analysed in the empirical part of the thesis.

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ABSTRACTS

The thesis ‘Voting paradoxes’ analyses paradoxes of power and the role of abstention in measures of power.

The first part aims at fundamental concepts of power and common power indices and their axioms. The importance and relevance of these axioms, as binding conditions of power measures, is considered profoundly.

The second section presents practically all known voting paradoxes with emphasise on the paradoxes of power and introduces a new paradox, the Paradox of abstention. As a result of its analysis, it is proved that the concept of global monotonicity is less binding than it is believed, provided committees where posterior abstention is allowed are analysed with the use of ‘binary’ indices of power, such as Shapley - Shubik or Banzhaf - Coleman are.

The last section deals with the role of abstention and introduces a new model of power index based on non - discriminatory ‘tripartition’. The empirical part uses this index for an analysis of the Chamber of Deputies of the Czech Republic.

ABSTRAKT

Diplomová práce ‘Volební paradoxy’ analyzuje paradoxy síly a roli absence při měření síly.

První část se zabývá základními koncepty, běžnými indexy volební síly a jejich axiomy. Zejména je zvážen význam a relevance těchto axiomů jakožto omezujících podmínek měření síly.

Druhá část představuje prakticky všechny volební paradoxy s důrazem na paradoxy síly a zavádí nový paradox, „paradox absence”. V důsledku jeho existence je dokázáno, že koncept globální monotonie je za předpokladu užití binárních indexů síly (jakými jsou například Shapley - Shubik nebo Banzhaf – Coleman) pro analýzy výborů, ve kterých je absence dovolena, méně restriktivní, než je obecně předpokládáno.

Závěrečná část práce se zabývá vlivem absence a zavádí nový model indexu síly založený na nediskriminující tripartitě. Empirická část užívá tento index pro analýzu Poslanecké sněmovny parlamentu České republiky.